$\boxed{\text{Jan 10}}$

- OH corrected MW 1:10 – 2:30
- This Wed (tomorrow): 1:10 – 2:00.
- No quiz, HW due this week. Both start next week.

Ex Return $n$ graded HW to $n$ students at random. What is the prob. that at least one ~~pers~~ student~~s~~ gets his/her own HW?

$A_i := \{$ i'th person gets ~~correct~~ right HW$\}$. Interested in

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \overset{IEP}{=} \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i_1<i_2} \mathbb{P}(A_{i_1} \cap A_{i_2}) + \dots$$

$$+ (-1)^{k-1} \sum_{i_1<\dots<i_k} \mathbb{P}(A_{i_1} \cap \dots \cap A_{i_k}) + \dots + (-1)^{n-1} \mathbb{P}(A_1 \cap \dots \cap A_n).$$

Now, $\mathbb{P}(A_i) = \frac{1}{n}$

$$\mathbb{P}(A_{i_1} \cap \dots A_{i_k}) = \boxed{\frac{(n-k)!}{n!}}$$

$\leftarrow$ # perm's fixing $\{i_1, \dots, i_k\}$

$\leftarrow$ total # perm's.

# summands in $\sum_{i_1<\dots<i_k}$ = # subsets of $k$ elements $= \binom{n}{k} = \boxed{\frac{n!}{k!(n-k)!}}$

$$\Rightarrow \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = 1 - \frac{1}{2!} + \frac{1}{3!} - \dots + \frac{(-1)^{k-1}}{k!} + \dots + \frac{(-1)^{n-1}}{n!}$$

$$\to 1 - \frac{1}{e} \approx 0.63 \quad \text{as } n \to \infty.$$

~~Week 10~~

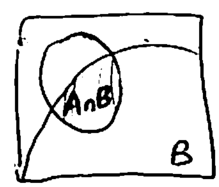
- OH ~~MW 1:10-2:30 pm~~

## 1.4. Conditional Probability

**Def** $A, B$ events, $P(B) > 0$.

$$\boxed{P(A \mid B) := \frac{P(A \cap B)}{P(B)}}$$

~~Motivation: B plays the role of a new "samp~~




**Ex** Community of 360 people.

| | Cancer | No cancer |
|---|---|---|
| Smoker | 8 | 32 |
| No smoker | 16 | 304 |

$$P(\text{Cancer}) = \frac{(8+16)}{360} \approx 0.07$$

$$P(\text{Cancer} \mid \text{Smokes}) = \frac{P(\text{Cancer} \cap \text{Smokes})}{P(\text{Smokes})} = \frac{8/360}{(8+32)/360} = \frac{8}{8+32} = 0.2$$

~~P(Cancer | not smokes)~~ - - - - - - - ~~0.05~~

Motivation for def: $B$ becomes a new "sample space" (e.g. "Smokes").

Consequence:

$$P(A \cap B) = P(A|B)\,P(B) \qquad (*)$$

Prop/Thm (**Law of Total Probability**) $\forall$ mutually exclusive (e.g. disjoint) events $B_1, B_2, \ldots$ (finite or countably $\infty$), such that

$$\Omega = \bigcup_i B_i,$$

$\forall$ event $A$,

$$P(A) = \sum_i P(A|B_i)\,P(B_i)$$

Pf: $A = \bigcup_i (A \cap B_i)$ mutually exclusive, $\Rightarrow$

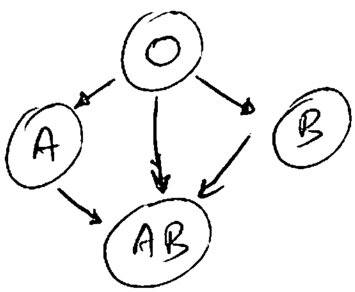
$$P(A) = \sum_i P(A \cap B_i) \overset{(*)}{=} \sum_i P(A|B_i)P(B_i).$$

Ex (Blood transfusion) Statistics of Blood types : ~~O, A, B, AB~~.

~~O (30%), A (20%)~~,

O (34%), A (38%), B (20%), AB (8%).

Blood transfusion chart (donor $\to$ recepient):




$P(\underbrace{\text{random donor can give blood to random recipient}}_{E}) = ?$

By conditioning on the blood type of the donor

$$P(E) = \underbrace{P(E|O)}_{\substack{\| \\ P(A)+P(B)+P(AB)}} \underset{0.34}{\overset{}{P(O)}} + \underbrace{P(E|A)}_{\substack{\| \\ P(A)+P(AB)}} \underset{0.38}{P(A)} + \underbrace{P(E|B)}_{\substack{\| \\ P(B)+P(AB)}} \underset{0.2}{P(B)} + \underbrace{P(E|AB)}_{\substack{\| \\ P(AB)}} \underset{0.08}{P(AB)}$$

$\simeq 0.57$.

"Reversing the order" of conditioning:

Prop 1.17 ~~Thm~~ (Bayes Rule) Let $B_1, B_2, \dots$ be mutually exclusive events, $\Omega = \bigcup_i B_i$,

$\forall$ event $A$:

$$P(B_j|A) = \frac{P(A|B_j)\,P(B_j)}{\sum_i P(A|B_i)\,P(B_i)}$$

$$P(B_j|A) = \frac{P(B_j \cap A)}{P(A)} \underset{LTP}{\overset{\text{P.12}}{=}} \frac{P(A|B_j)\,P(B_j)}{\sum_i P(A|B_i)\,P(B_i)}.$$

Ex A certain disease affects 1% of population.
A test is false positive in 5% cases, false negative in 10% cases.
$\uparrow P(P|D^c)$ $\quad P(P^c|D)$
A person is tested positive. ~~P(sick~~ Prob. of ~~sick?~~ having the disease?

$D := \{\text{having disease}\}$, $P := \{\text{test positive}\}$ ~~[illegible]~~.

$$P\{D|P\} = \frac{P(P|D)\,P(D)}{\underbrace{P(P|D)}_{0.9}\underbrace{P(D)}_{0.01} + \underbrace{P(P|D^c)}_{0.05}\underbrace{P(D^c)}_{0.99}} = 0.18. \quad \text{Low!}$$

## 1.5. Independence

**Def** Events $A, B$ are independent if
$$\boxed{P(A \cap B) = P(A)\,P(B)}$$
$$\Updownarrow$$

Motivation: $P(A|B)\,P(\cancel{B}) = P(A)\,P(\cancel{B})$.

$$\boxed{P(A|B) = P(A)}$$

Similarly, $\boxed{P(B|A) = P(B)}$

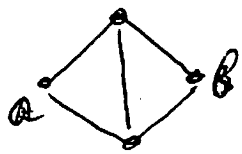
**Def.** A finite family of events $A_1, \dots, A_n$ is independent if $\forall$ subset $I \subset \{1, \dots, n\}$,
$$P\Big(\bigcap_{i \in I} A_i\Big) = \prod_{i \in I} P(A_i).$$

- family $\{A_\alpha : \alpha \in I\}$ is independent if $\forall$ finite sub-family is.

**Warning**: Pairwise independence (for $|I| = 2$ only) does not imply (full) independence.

**Ex**: roll 3 dice.
$A = \{\text{first die} = 3\}$, $B = \{\text{second die} = 4\}$, $C = \{\text{sum of all three} = 7\}$.
Check that $A, B$ are indep; $B, C$ indep, $A, C$ indep, but $A, B, C$ not.

Ex (Networks). Consider the network




Each link can fail independently with prob. $p$.

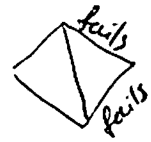
Prob. $\underbrace{(a, b \text{ connected})}_{\parallel\atop C} = ?$

Condition on the state of the vertical link. $V := \{\text{vert. link works}\}$.

~~$P(C|V) = P(C \cap$~~

$$P(C) = P(C|V)\underbrace{P(V)}_{\parallel \atop 1-p} + P(C|V^c)\underbrace{P(V^c)}_{\parallel \atop p}$$

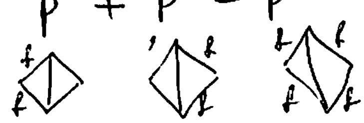
(1) $P(C|V) = ?$

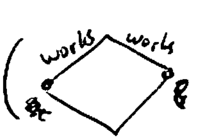
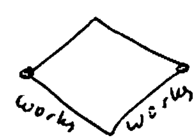
$P(C^c|V) = \{$  or  $\}$ (or both).



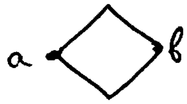
$\neq$ prob $= p^2$ by indep; prob $= p^2$ by indep.

$= p^2 + p^2 - p^4$ (by inclusion-exclusion)




(2) $P(C|V^c) = P\Big($  or  $\Big)$ (or both)






$= (1-p)^2 + (1-p)^2 - (1-p)^4$ (by incl - excl)

Hence
$$P(C) = (1 - 2p^2 + p^4)(1-p) + \left[2(1-p)^2 - (1-p)^4\right]p$$
$$= 1 - 2p^2 - 2p^3 + 5p^4 - 2p^5.$$