Example (Simple random walk)

A particle is placed at \( k \).
Each second, the particle moves 1 step to the left or to the right independently with prob \( \frac{1}{2} \) each.
What is the probability that the particle reaches \( n \) before reaching \( 0 \)?

Condition on the first step, \( L \) or \( R \)

\[
P(E_k) = P(E_k | L) P(L) + P(E_k | R) P(R).
\]

\[
= P(E_{k-1}) \cdot \frac{1}{2} + P(E_{k+1}) \cdot \frac{1}{2}
\]

(Conditioned on \( L \), the process "renews".)

(Will particle at \( k-1 \) instead of \( k \))

Denoting \( P_k = P(E_k) \), we obtain

\[
\begin{cases}
P_k = \frac{1}{2} (P_{k-1} + P_{k+1}), & k = 1, \ldots, n-1 \\
P_0 = 0, & P_n = 1
\end{cases}
\]

\( n+1 \) linear equations with \( n+1 \) unknowns. Solving (do this!) gets us

\[
P_k = \frac{k}{n}
\]

(obviously this is a solution)

• Interpretations:

(a) Finance: \( 0 = \) bankruptcy, \( n = \) payoff, \( k = \) initial capital

\( P(\text{payoff before bankruptcy}) \)?

For this example, a biased random walk is more relevant (see Section 1.7)

where \( P(R) = p \), \( P(L) = 1 - p \) for some \( 0 < p < 1 \)

"Non-symmetric random walk"

(b) \( k \)-renewal: \( \) \( n \to \infty \), \( P_k \to 0 \) (with \( k \) fixed)

\( \) with prob. 1 ("almost surely"), the particle will visit any given site (on \( k \) this row)

(If last, will return home by random walk)
2.1. Random Variables.

- Intuitively, a random variable = numerical values associated with the outcomes of a random experiment.

Ex: \(|\Omega| = \text{flip 3 coins, record # of heads}

\[ \Omega = \{TTT, TTH, THT, THH, HTT, HTH, HHT, KKT, KKH, KTH\} \]

- Lifetime of a phone you are going to buy
- Flight delay

Def \((\Omega, \Sigma, \mathbb{P})\) be a prob. space.

A random variable is a function \(X: \Omega \rightarrow \mathbb{R}\) such that 

\[ \forall a \in \mathbb{R}, \{\omega : X(\omega) \leq a\} \text{ is an event.} \quad (\text{i.e. } \in \Sigma). \]

Ex \(\{\omega : X(\omega) \leq \# 1\} = \{\text{at most 1 head}\} = \{TTT, TTH, THT, THH\} \).

Convention: \(\{\omega : X(\omega) \leq a^2\} = \{X \leq a\}. \quad \|

Prop \forall \text{ r.v. } X, \text{ the following are events: } \forall a, b \in \mathbb{R}:

(i) \(\{X > a\}\)

(ii) \(\{X < a\}\) and \(\{X \geq a\}\)

(iii) \(\{a < X \leq b\}\), \(\{a \leq X < b\}\), etc.

\[ (i) \{X > a\} = \left\{ \left. x \in \mathbb{R} \right| a < x \right\} \in \Sigma \text{ by an axiom of } \sigma\text{-algebra.} \]

\[ (ii) \{X < a\} = \bigcup_{b \in \mathbb{Q} \cap (0, a)} \{X < b\}. \quad \text{"}\bigcup\text{" is obvious: } X < b \text{ implies } X < a. \]

\[ (iii) \text{ Each } \{X < b\} \text{ is an event.} \rightarrow X(\omega) < a \rightarrow \exists \theta \in \mathbb{Q} : \quad X(\omega)(\theta) < b \rightarrow \theta = a \]

(iii) ... Countable intersection of events is an event (axiom of } \sigma\text{-alg.).} \]
Prop. (Ex. 2.1). Let \( X, Y \) be r.v.s and \( a \in \mathbb{R} \).

Then the following are r.v.s:

(i) \( aX \);
(ii) \( X + Y \);
(iii) \( XY \);
(iv) \( Z : = \begin{cases} Y/X & \text{if } X \neq 0 \\ 0 & \text{if } X = 0. \end{cases} \)

Fix \( V \subset \mathbb{R} \).Enough to show that
\[ \{ X + Y > a \} \] is an event.

\[ \{ X + Y > a \} = \bigcup_{q \in \mathbb{Q}} \{ X > q, Y > a - q \} \]

Check!

an event (intersections of 2 events)

an event (countable intersection of events)

Ex. (Indicator) An indicator of an event \( E \) is a r.v.

\[ 1_E : = \begin{cases} 1 & \text{if } E \text{ occurs} \\ 0 & \text{if } E \text{ does not occur}. \end{cases} \]

Formally:

\[ 1_E (w) : = \begin{cases} 1 & \text{if } w \in E \\ 0 & \text{if } w \notin E. \end{cases} \]

\( 1_E \) is a r.v! (Check!)
Cumulative Distribution Functions (CDF).

Def: Let $X$ be a r.v. The distribution $F$ of $X$ is a function $F: \mathbb{R} \to [0, 1]$ defined as:

$$F(a) = P\{X \leq a\}, \quad a \in \mathbb{R}.$$

Ex. 1. $X = \#$ heads in 3 coin flips (p. 17).

$$F(a) = \begin{cases} 
0, & a < 0 \quad (P\{X < 0\} = 0) \\
\frac{1}{8}, & 0 \leq a < 1 \quad (P\{1 \leq a < 2\} = \frac{1}{8}) \\
\frac{1}{2}, & 1 \leq a < 2 \\
\frac{7}{8}, & 2 \leq a < 3 \\
1, & a \geq 3
\end{cases}$$

Ex. 2. $X =$ time I walk out of my house. Any time between 8-9 am is equally likely. ($X$ "uniformly dist. in [8, 9]")

$$F(a) = P\{X \leq a\} = \begin{cases} 
\frac{a-8}{9-8} = a-8, & a \in [8, 9]
\end{cases}$$

Terminology: The distribution of $X$ is the complete information of what values $X$ takes with what probabilities.

Ex. 3. Throw a dart into the circle of radius 1; any pt is equally likely to be hit. ($X$ "uniformly dist. in circle").

$$X = \text{dist. to center.}$$

$$F(a) = P\{X \leq a\} = \frac{\text{Area (Circle radius } a\text{)}}{\text{Area (Circle radius 1)}} = \frac{a^2}{\pi} = a^2, \quad a \in (0, 1).$$
Prop. 2.5 Let \( F \) be the CDF of a r.v. \( X \). Then \( F \) is

(i) monotonically increasing: \( A \subseteq B \) implies \( F(a) \leq F(b) \);

(ii) right-continuous: \( \lim_{x \to a^{+}} F(x) = F(a) \forall a \in \mathbb{R} \).

(iii) \( \lim_{x \to -\infty} F(x) = 0 \) and \( \lim_{x \to +\infty} F(x) = 1 \).

(i) \( F(a) = \Pr\{X \leq a\} \leq \Pr\{X \leq b\} = F(b) \).

(ii) Let's check the sequential def. of right-continuity:

\[ \forall x_{n} \uparrow a : \lim_{n \to \infty} F(x_{n}) = F(a). \]

\( B_{n} := \{x \leq x_{n}\} \Rightarrow B_{1} \supseteq B_{2} \supseteq B_{3} \ldots \)

By continuity of probability (Prop. 2.6, p.7),

\[ \lim_{n \to \infty} \Pr(B_{n}) = \Pr(B) \quad \text{where} \quad B = \bigcap_{n=1}^{\infty} B_{n}. \]

\[ F(x_{n}) \text{ by def.} \quad \Pr\{ X \leq x_{n} \forall n \} = \Pr\{ X \leq a \} = F(a). \]

QED.

(iii) \( \lim_{x \to +\infty} F(x) = 1 ? \) Check sequential def:

\[ \forall x_{n} \uparrow \infty : \lim_{n \to \infty} F(x_{n}) = 1. \]

\( A_{n} := \{ x \leq x_{n} \} \Rightarrow A_{1} \supseteq A_{2} \supseteq A_{3} \ldots \)

By continuity of prob (Prop. p.7),

\[ \lim_{n \to \infty} \Pr(A_{n}) = \Pr(A). \]

\[ A = \bigcup_{n=1}^{\infty} A_{n} = \{ \exists x_{n} : X \leq x_{n} \} = \Omega \quad \text{(since } x_{n} \to \infty). \]

\[ \Rightarrow \Pr(A) = 1. \]

Cor: \( F \) has at most a countable # of discontinuities.

(As is a monotone function on \( \mathbb{R} \), see real analysis).