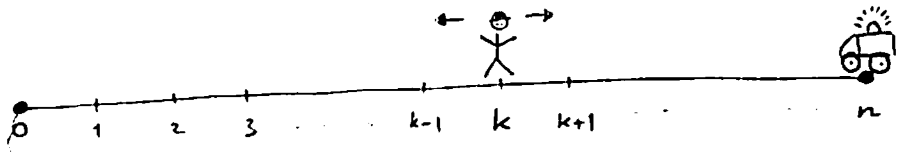
Jan 12 ~~Quiz (everything except this class)~~. Book: reserve, e-form.

1.7 Example (Simple random walk)



A particle is placed at k.
Each second, the particle moves 1 step to the left or to the right independently with prob $\frac{1}{2}$ each.
What is the probability that $\underbrace{\text{the particle reaches } n \text{ before reaching } 0}_{E_k}$?

Condition on the first step, L or R

$$P(E_k) = P(E_k \mid L)\,P(L) + P(E_k \mid R)\,P(R)$$
$$= P(E_{k-1})\cdot\frac{1}{2} + P(E_{k+1})\cdot\frac{1}{2}$$

(Conditioned on L, the ~~game~~ process "resets" with particle at $k-1$ instead of k)

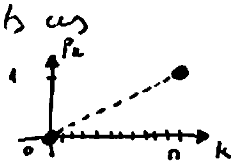
Denoting $p_k = P(E_k)$, we obtain

$$\begin{cases} p_k = \frac{1}{2}(p_{k-1} + p_{k+1}), & k = 1, \dots, n-1 \\ p_0 = 0,\ p_n = 1 \end{cases}$$

$n+1$ linear equations with $n+1$ unknowns. Solving (do this!) gets us

$$\boxed{p_k = \frac{k}{n}}$$

(obviously this is a solution)



- <u>Interpretations</u>:

(a) <u>Finance</u>: 0 = bankrupcy, n = payoff, k = initial capital
P(payoff before bankrupcy) = ?

For this example, a <u>biased</u> random walk is more relevant, where $P(R) = p$, $P(L) = 1-p$ for some $0<p<1$ (See ~~Ross Ex 4l~~ Section 1.7)
"NON-SYMMETRIC RANDOM WALK"

~~(b) Recurrence: for $n \to \infty$, $p_k \to 0$ (with k fixed)~~
~~$\Rightarrow$ with prob. 1 ("almost surely"), the particle will visit any given site (0 in this case)~~
~~(If lost, will return home by random walk)~~

2.1. Random variables.

- *Intuitively*, a random variable = numerical values associated with the outcomes of a random experiment.

Ex: • flip 3 coins, record # of heads

$$\begin{array}{ccccccccc} \Omega = \{ & TTT, & TTH, & THT, & THH, & HTT, & HTH, & HHT, & HHH\} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 \end{array}$$

- lifetime of a phone you are going to buy
- Flight delay

Def Let (Ω, Σ, P) be a prob. space.
A *random variable* is a function $X : \Omega \to \mathbb{R}$ such that $\forall a \in \mathbb{R}$, $\{\omega : X(\omega) \le a\}$ is an event. (i.e. $\in \Sigma$).

Ex $\{\omega : X(\omega) \le 1\} = \{\text{at most 1 head}\} = \{TTT, TTH, THT, HTT\}$.

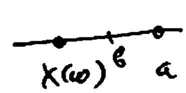
Ex $P\{X \le 1\} = \frac{4}{8} = \frac{1}{2}$.

Convention: $\boxed{\{\omega : X(\omega) \le a\} = \{X \le a\}.}$

Prop $\forall$ r.v. X, the following are events: $\forall a, b \in \mathbb{R}$:
(i) $\{X > a\}$
(ii) $\{X < a\}$ and $\{X \ge a\}$
(iii) $\{a < X < b\}$, $\{a \le X < b\}$, etc.

(i) $\{X > a\} = \underbrace{\{X \le a\}}_{\in \Sigma}{}^c \in \Sigma$ by an axiom of σ-algebra.

(ii) $\{X < a\} = \bigcup_{\substack{b \in \mathbb{Q} \\ b < a}} \{X < b\}$ "$\supset$" is obvious: $X < b$ implies $X < a$.
"$\subset$": $\forall \omega : X(\omega) < a$. $\exists\, b \in \mathbb{Q}$: $X(\omega) < b < a \Rightarrow \square$



Each $\{X < b\}$ is an event.
Countable intersection of events is an event (axiom of σ-alg.).

(iii) ...

(OPERATIONS ON R.V'S)

<u>Prop.</u> (Ex. 2.1). Let X, Y be r.v's, and $a \in \mathbb{R}$.

Then the following are r.v's:

(i) aX;

(ii) $X+Y$;

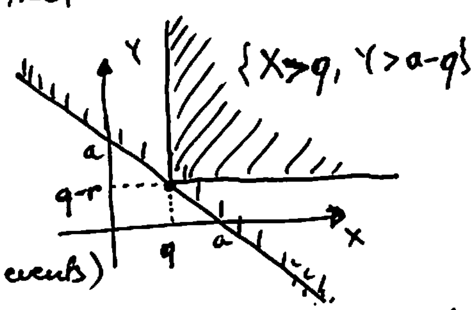
(iii) XY;

(iv) $Z := \begin{cases} Y/X & \text{if } X \neq 0 \\ 0 & \text{if } X = 0. \end{cases}$

(ii) Fix $\forall a \in \mathbb{R}$. Enough to show that $\{X+Y>a\}$ is an event.

$$\{X+Y>a\} = \underbrace{\bigcup_{q\in\mathbb{Q}} \underbrace{\{X>q,\ Y>a-q\}}_{\text{an event (intersections of 2 events)}}}_{\text{an event (countable intersection of events)}}$$

Check! $\longrightarrow$



<u>Ex</u> (<u>Indicator</u>) An <u>indicator</u> of an event E is a r.v.

$$\mathbb{1}_E := \begin{cases} 1 & \text{if } E \text{ occurs} \\ 0 & \text{if } E \text{ does not occur.} \end{cases}$$

(Formally: $\mathbb{1}_E(\omega) := \begin{cases} 1 & \text{if } \omega \in E \\ 0 & \text{if } \omega \notin E \end{cases}$)

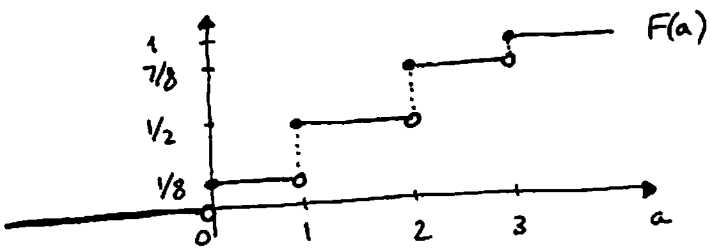
$\mathbb{1}_E$ is a r.v! (Check!)

Cumulative Distribution functions. (CDF).

Def Let X be a r.v. ~~The distribut~~ The CDF of X is a function $F: \mathbb{R} \to [0,1]$ defined as

$$F(a) := \mathbb{P}\{X \le a\}, \quad a \in \mathbb{R}.$$

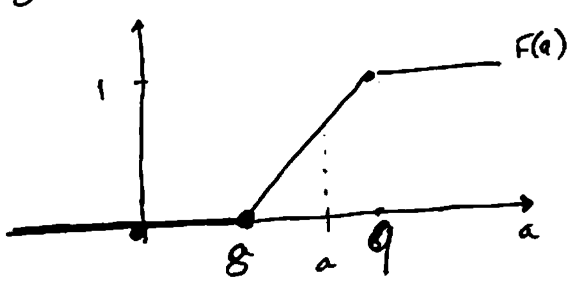
Ex 1. X = # heads in 3 coin flips (p.17).



$$F(a) = \begin{cases} 0, & a < 0 \quad (\mathbb{P}\{X \le a\} = 0) \\ 1/8, & 0 \le a < 1 \quad (\mathbb{P}\{X \le a\} = \mathbb{P}\{TTT\} = \frac{1}{8}) \\ 1/2, & 1 \le a < 2 \quad \ldots \\ 7/8, & 2 \le a < 3 \\ 1, & a \ge 3 \end{cases}$$

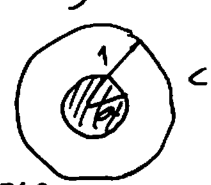
Ex. 2 X = time I walk out of my house.
Any time btw 8-9 am is equally likely. (X "uniformly distr. in $[8,9]$")

$$F(a) = \mathbb{P}\{X \le a\} = \frac{a-8}{9-8} = a-8, \quad a \in [8,9]$$

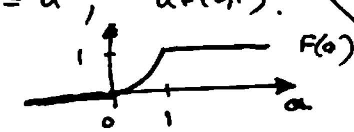


~~Terminology The distribution of X is the ~~knowledge complete~~ ~~information about~~ knowledge of what values X takes ~~with~~ what probabilities.~~

Ex. 3. Throw a dart into the circle of radius 1; ~~all~~ any pt is equally likely to be hit. (X "uniformly distr. in circle").
$X :=$ dist. to center.



$$F(a) = \mathbb{P}\{X \le a\} = \frac{\text{Area (Circle radius } a)}{\text{Area (Circle radius } 1)} = a^2, \quad a \in (0,1).$$



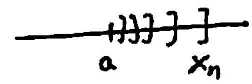
(Properties of CDF)

$$\boxed{\begin{array}{l}\textbf{Prop. 2.5} \text{ Let } F \text{ be the CDF of a r.v. } X. \text{ Then } F \text{ is} \\ \text{(i) monotonically increasing: } a \le b \text{ implies } F(a) \le F(b); \\ \text{(ii) right-continuous: } \lim_{x \to a_+} F(x) = F(a) \quad \forall a \in \mathbb{R}. \\ \text{(iii) } \lim_{x \to -\infty} F(x) = 0 \text{ and } \lim_{x \to +\infty} F(x) = 1. \end{array}}$$

(i) $F(a) = \mathbb{P}\{\underline{X \le a}\} \le \mathbb{P}\{\underline{X \le b}\} = F(b)$.

$\{X \le a\} \subseteq \{X \le b\}$

(ii) Let's check the sequential def. of right-continuity:

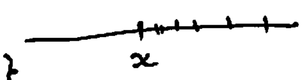


$$\forall x_n \downarrow a : \lim_{n\to\infty} F(x_n) = F(a).$$

$B_n := \{X \le x_n\}$. $\Rightarrow$ $B_1 \supseteq B_2 \supset B_3 \supset \cdots$

By continuity of probability (Prop. p.7),

$$\lim_{n\to\infty} \underbrace{\mathbb{P}(B_n)}_{\substack{\| \\ F(x_n) \text{ by def.}}} = \underbrace{\mathbb{P}(B)}_{\substack{\| \\ \mathbb{P}\{\underbrace{X \le x_n \ \forall n}_{X \le x}\} = \mathbb{P}\{X \le x\} = F(x).}} \quad \text{where } B = \bigcap_{n=1}^{\infty} B_n.$$



QED

(iii) $\lim_{x\to+\infty} F(x) = 1$? Check sequential def:

$$\forall x_n \uparrow \infty : \lim_{n\to\infty} F(x_n) = 1.$$

~~$B_n = \{X \le x_n\}$.~~

$A_n := \{X \le x_n\}$. $\Rightarrow$ $A_1 \subset A_2 \subset A_3 \subset \cdots$

By continuity of prob. (Prop. p.7),

$$\lim_{n\to\infty} \underbrace{\mathbb{P}(A_n)}_{\substack{\| \\ F(x_n) \text{ by def.}}} = \mathbb{P}(A). \qquad A = \bigcup_{n=1}^{\infty} A_n = \{\exists x_n : X_n \le n\}$$

$$= \Omega \quad (\text{since } x_n \to \infty).$$

$$\Rightarrow \mathbb{P}(A) = 1.$$

$\boxed{\textbf{Cor} \ F \text{ has at most a countable \# of discontinuities}}$

(As is $\forall$ monotone function on $\mathbb{R}$ – see real analysis).