Jan 17 Quiz: covers same material as HW (1.1–1.6). Midterms: I (Feb 16)
II (March 23)

<u>Prop. 2.6</u> (More properties of CDF)

Let F be the CDF of a r.v. X. Then $\forall\, a<b$:

(i) $\mathbb{P}\{X<a\} = \lim_{x\to a-} F(x) \quad (=: F(a_-).)$ ~~$\forall a\in\mathbb{R}$~~

(ii) $\mathbb{P}\{X=a\} = F(a) - F(a_-)$

(iii) $\mathbb{P}\{a<X\le b\} = F(b) - F(a)$

(iv) $\mathbb{P}\{X>a\} = 1-F(a)$

(i) $\underbrace{\{X<a\}}_{\overset{\shortparallel}{A}} = \bigcup_{n=1}^{\infty} \underbrace{\{X \le a-\tfrac{1}{n}\}}_{\overset{\shortparallel}{A_n}}$

$a-\frac{1}{n}$ a

Then $A_1 \subset A_2 \subset A_3 \subset \cdots$, $A = \bigcup_{n=1}^{\infty} A$

$\Rightarrow$ by continuity of prop. (p. 7),

$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$. D.

(ii) $\mathbb{P}\{X=a\} = \mathbb{P}\{X\le a\} - \mathbb{P}\{X<a\}$ (Why?)

$= F(a) - F(a_-)$ (by def. & (i)).

<u>Remark</u> The CDF F of X allows us to calculate
$\mathbb{P}\{X\in A\}$ for many subsets $A\subset\mathbb{R}$:

- $\forall$ <u>interval</u> A, (see Prop. above),
- $\forall$ <u>countable union</u> of intervals: if $A = \bigcup_n A_n$
  then $\mathbb{P}\{X\in A\} = \sum_n \mathbb{P}\{X\in A_n\}$
- $\forall$ <u>Borel set</u> A (formed from intervals using countable unions, complements)

<u>Remark</u> · Borel subsets of $\mathbb{R}$ form a σ-algebra.
· Borel set = "$\forall$ set of potential interest".

2.2. Existence of r.v.'s.

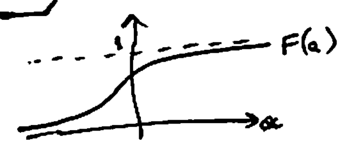
Thm. 2.14 Let $F: \mathbb{R} \to \mathbb{R}$ be a function that is

(i) monotonically increasing;

(ii) right-continuous;

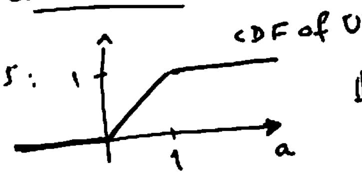
(iii) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$.

Then $\exists$ r.v. X whose CDF is F.



(In the partial case where F is continuous):

Let $U \sim \mathrm{Unif}\,[0,1]$, which means:



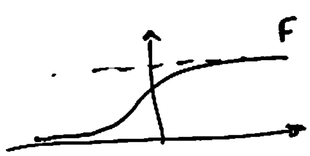
$\mathbb{P}\{U \le a\} = a, \quad a \in [0,1]$

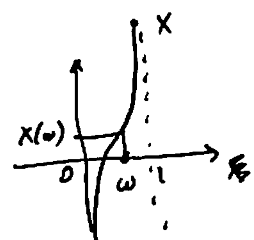
$X := F^{-1}(U)$.

Claim that CDF of X is F. Proof:

$\mathbb{P}\{X \le a\} = \mathbb{P}\{F^{-1}(U) \le a\} = \mathbb{P}\{U \le F(a)\} = a$. QED.

(Picture:





).

Remark: general case (right-continuous) — see book.

2.3. Independence of r.v's.

· Informally: X, Y indep. $\Leftrightarrow$ take values independently $\Leftrightarrow$ $\{X \in A\}, \{Y \in B\}$ are indep $\forall A, B \subset \mathbb{R}$.

Ex · Exit times · Not height, weight.
· Temp, stock value

$\Leftrightarrow \mathbb{P}\{X \le x, Y \le y\} = \mathbb{P}\{X \le x\} \cdot \mathbb{P}\{Y \le y\}$.

Def · R.v's X, Y are *independent* if the events

$$\{X \le x\} \text{ and } \{Y \le y\}$$

are independent $\forall x, y \in \mathbb{R}$.

· R.v's $X_1, \dots, X_n$ are *independent* if the events

$$\{X_1 \le x_1\}, \dots, \{X_n \le x_n\}$$

are independent $\forall x_1, \dots, x_n \in \mathbb{R}$.

· A family $\{X_\alpha : \alpha \in I\}$ is *independent* if $\forall$ finite sub-family is.

Prop X, Y independent $\Rightarrow$ the events

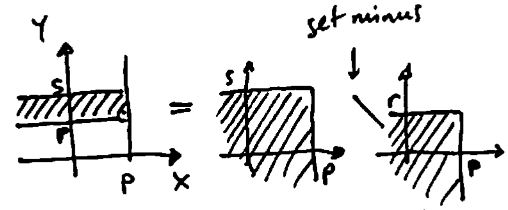
$$\{X \in A\} \text{ and } \{Y \in B\}$$

are independent $\forall$ intervals ~~All sets of the A=(−∞,~~ of the form $A = (p, q]$, $B = (r, s]$.

$\mathbb{P}\{X \in A, Y \in B\} = \mathbb{P}\{X \in A\} \cdot \mathbb{P}\{Y \in B\}$?

· First, for $I_q := (-\infty, q]$ instead of A.

~~P{X∈(−∞,q], Y∈B~~



$$\mathbb{P}\{X \in (-\infty, q], Y \in (r, s]\} = \mathbb{P}\{X \le q, r < Y \le s\}$$

$$= \mathbb{P}\{X \le q, Y \le s\} - \mathbb{P}\{X \le q, Y \le r\}$$

$$= \mathbb{P}\{X \le q\} \mathbb{P}\{Y \le s\} - \mathbb{P}\{X \le q\} \cdot \mathbb{P}\{Y \le r\} \quad \text{(by independence)}$$

$$= \mathbb{P}\{X \le q\} \left(\mathbb{P}\{Y \le s\} - \mathbb{P}\{Y \le r\} \right)$$

$$= \mathbb{P}\{X \le q\} \cdot \mathbb{P}\{r < Y \le s\}$$

$$= \mathbb{P}\{X \in (-\infty, q]\} \cdot \mathbb{P}\{Y \in (r, s]\}. \qquad \square$$

· Now, repeat this reasoning ~~to get~~ for X to get

$$\mathbb{P}\{X \in (p, q], Y \in (r, s]\} = \dots = \mathbb{P}\{X \in (p, q]\} \cdot \mathbb{P}\{Y \in (r, s]\}.$$

Remark Extends further to $\forall$ Borel sets A, B.

2.4. Types of Distributions.

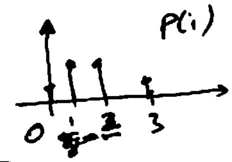
Def X has a <u>discrete</u> distribution if X takes on finite or countably infinitely many values $\{x_i\}$.

$$\mathbb{P}\{X = x_i\} = p_i, \qquad 0 < p_i \le 1, \quad \sum_i p_i = 1.$$

$p(i) = p_i$ is called the <u>probability mass function</u> (PMF).

Ex. (# heads in 3 tosses)

$$p(0) = \frac{1}{8}, \quad p(1) = \frac{3}{8}, \quad p(2) = \frac{3}{8}, \quad p(3) = \frac{1}{8}.$$



PMF ⟷ PDF : $p(i) = F(x_i) - F(x_i-)$;

Def X has an (absolutely) <u>continuous</u> distribution if

$\exists\, f : \mathbb{R} \to \mathbb{R}$ s.t.

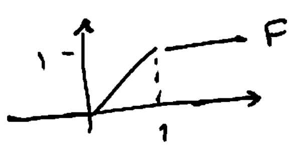
$$F(\alpha) = \int_{-\infty}^{\alpha} f(x)\,dx. \qquad \forall\, \alpha \in \mathbb{R}$$

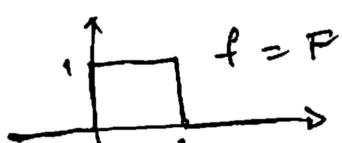
↑ CDF of X This f is called the <u>density of X</u> (PDF of X)

⇕

F is differentiable (at all pts of continuity of f) on $\mathbb{R}$); $F'(a) = f(a)$.

Ex ① $X \sim \mathrm{Unif}\,[0,1]$





∀pts equally likely

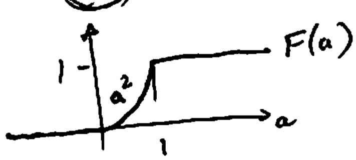
③ $X \sim N(0,1)$ "<u>standard normal</u>"

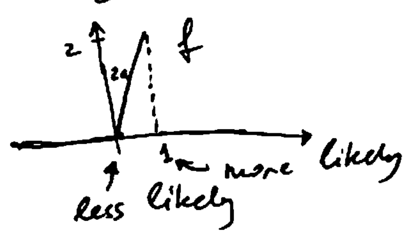
$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \quad x \in \mathbb{R}$$

$$F(\alpha) = \int_{-\infty}^{\alpha} f(x)\,dx =: \Phi(x) \quad \left(\begin{matrix}\text{Related to} \\ \text{erf}\end{matrix}\right)$$

Remarks.

② Distance to the center of the target (Ex. p.19):





Meaning of density: $\mathbb{P}\{X \in [x, x+\varepsilon]\} = F(x+\varepsilon) - F(x) \approx f(x)\cdot\varepsilon$.