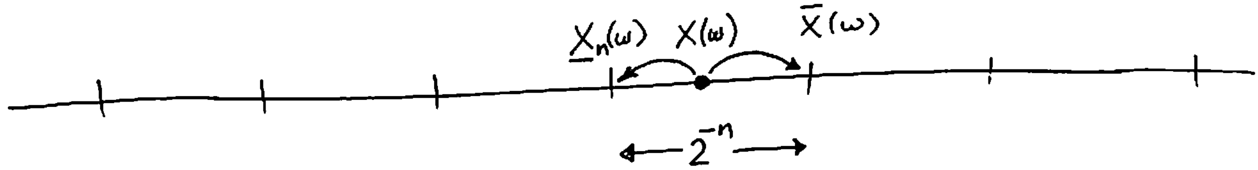
$\boxed{\text{Jan 31}}$

3.1. Expectations of general r.v's.

(In particular, continuous – but not necessarily).

By approximation with discrete:

• Let X be a r.v., ~~$n \in \mathbb{N}$~~ $n = 0, 1, 2, \dots$



Def Define lower & upper dyadic approximations $\underline{X}_n, \overline{X}_n$ as follows: ~~r.v's~~

if $\dfrac{k}{2^n} < X(\omega) \le \dfrac{k+1}{2^n}$

set $\underline{X}_n(\omega) := \dfrac{k}{2^n}$, $\overline{X}_n(\omega) := \dfrac{k+1}{2^n}$.

Elementary properties:

(i) $\underline{X}_n$, $\overline{X}_n$ are discrete r.v's

(ii) $\underline{X}_n(\omega) < X(\omega) \le \overline{X}_n(\omega)$.

(iii) $\overline{X}_n(\omega) = \underline{X}_n(\omega) + 2^{-n}$

(iv) $\underline{X}_0(\omega) \le \underline{X}_1(\omega) \le \underline{X}_2(\omega) \le \dots \le X(\omega)$; $\overline{X}_0(\omega) \ge \overline{X}_1(\omega) \ge \overline{X}_2(\omega) \ge \dots \ge X(\omega)$.

(iv) $\xRightarrow{\text{MCT}}$ $\exists \lim_{n\to\infty} \underline{X}_n(\omega) = \lim_{n\to\infty} \overline{X}_n(\omega) = X(\omega)$

(vi) $\overline{X}_n$ & $\underline{X}_n$ are either all have expectation, or they all fail to have exp.

(vii) In case they do, (iv) implies $\mathbb{E}[\underline{X}_0] \le \mathbb{E}[\underline{X}_1] \le \mathbb{E}[\underline{X}_2] \le \dots \le \mathbb{E}[\overline{X}_2] \le \mathbb{E}[\overline{X}_1] \le \mathbb{E}[\overline{X}_0]$;

$$\mathbb{E}[\overline{X}_n] - \mathbb{E}[\underline{X}_n] = 2^{-n}$$

Def Let X be a (general) r.v.
We say that X has *expectation* if one of $\bar{X}_n, \underline{X}_n$ (and thus all of them) has expectation; and we define
$$\mathbb{E}[X] := \underbrace{\lim_{n\to\infty} \mathbb{E}[\bar{X}_n] = \lim_{n\to\infty} \mathbb{E}[\underline{X}_n]}_{\text{same by property (vii).}}$$

All basic properties ~~that we pr~~ of $\mathbb{E}$ that we proved for discrete r.v's X hold for general X, too. For example:

Thm (linearity) If r.v.'s X, Y have exp, then $X+Y$ does, too;
(i) $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
(ii) $\mathbb{E}[aX] = a\cdot\mathbb{E}[X]$.

(i) By reduction to discrete r.v's $\bar{X}_n, \bar{Y}_n$.

- $Z := X+Y$. $\bar{Z}_n \neq \bar{X}_n + \bar{Y}_n$ in general, but is close.

$$|\bar{Z}_n - \bar{X}_n - \bar{Y}_n| \le \underbrace{|Z-X-Y|}_{=0} + \underbrace{|\bar{Z}_n - Z|}_{\le 2^{-n}} + \underbrace{|\bar{X}_n - X|}_{\le 2^{-n}} + \underbrace{|\bar{Y}_n - Y|}_{\le 2^{-n}} \le 3\cdot 2^{-n}. \quad (*)$$

- (Existence of exp)
  To show Z has expectation, we bound
  $$|\bar{Z}_n| \le |\bar{X}_n| + |\bar{Y}_n| + 3\cdot 2^{-n} \quad \text{(by * \& } \Delta \text{ ineq.)}$$
  RHS has expectation since $\mathbb{E}|\bar{X}_n| < \infty$, $\mathbb{E}|\bar{Y}_n| < \infty$.
  $\xrightarrow{\text{prop}}$ $\mathbb{E}|\bar{Z}_n| < \infty \Rightarrow Z$ has exp

- (Identity) Using $(*)$ again, we see that
  $$\left|\underbrace{\mathbb{E}[\bar{Z}_n - \bar{X}_n - \bar{Y}_n]}_{=\mathbb{E}[\bar{Z}_n] - \mathbb{E}[\bar{X}_n] - \mathbb{E}[\bar{Y}_n]}\right| \le 3\cdot 2^{-n} \to 0$$
  (use property $|T|<a \Rightarrow \mathbb{E}|T| \le a$ for discrete r.v's)
  (linearity for discrete r.v's)
  $$\Rightarrow \mathbb{E}[Z_n] - \mathbb{E}[\bar{X}_n] - \mathbb{E}[\bar{Y}_n] \to 0$$
  $$\Rightarrow \lim_n \mathbb{E}[Z_n] = \lim_n \mathbb{E}[\bar{X}_n] + \lim_n \mathbb{E}[\bar{Y}_n] \quad \text{(linearity of limit)}$$
  $$\Rightarrow \mathbb{E}[Z] = \mathbb{E}[X] + \mathbb{E}[Y]. \quad \square$$

$\boxed{\text{THM 3.5 If } X \text{ and } Y \text{ are } \underline{\text{independent}} \text{ r.v's (with general distr.'s) that have exp's then } XY \text{ has exp, and } \mathbb{E}(XY) = \mathbb{E}[X]\cdot\mathbb{E}[Y].}$

Again, by reduction to discrete r.v's $\bar{X}_n$, $\bar{Y}_n$.

$$
\begin{aligned}
|XY - X_n Y_n| &= |XY - X\bar{Y}_n + X\bar{Y}_n - \bar{X}_n\bar{Y}_n| \\
&= |X(Y-\bar{Y}_n) + (X-\bar{X}_n)Y_n| \\
&\overset{\triangle}{\le} |X|\cdot\underbrace{|Y-\bar{Y}_n|}_{\le\, 2^{-n}} + \underbrace{|X-\bar{X}_n|}_{\le\, 2^{-n}}\cdot\underbrace{|\bar{Y}_n|}_{\le\, |Y| + |\bar{Y}_n - Y| \,\le\, |Y| + 2^{-n}} \\
&\le 2^{-n}|X| + 2^{-n}\left(|Y| + 2^{-n}\right) =: R_n \quad \text{(r.v).} \qquad (*)
\end{aligned}
$$

- (<u>Existence of exp</u>): By $(*)$ and $\triangle$,

$$|XY| \le \underbrace{|X_n|}\cdot\underbrace{|Y_n|} + \underbrace{R_n}$$

have exp's. $\Rightarrow$ (by discrete Thm) RHS has exp,

$\Rightarrow$ LHS has exp.

- (<u>Identity</u>) By $(*)$ again,

$$\underbrace{\left|\mathbb{E}[XY - X_nY_n]\right|}_{\mathbb{E}[XY] - \mathbb{E}[X_n]\cdot\mathbb{E}[Y_n]} \le \mathbb{E}R_n \to 0$$

(use property: $U \le V$ pointwise $\Rightarrow \mathbb{E}[U] \le \mathbb{E}[V]$ for discrete r.v's)

(linearity & Thm for discrete)

$$\Rightarrow \mathbb{E}[XY] = \lim_n \mathbb{E}[X_n]\cdot\mathbb{E}[Y_n] = \lim_n \mathbb{E}[X_n]\cdot\lim_n \mathbb{E}[Y_n] \quad \text{(prop. of limit)}$$

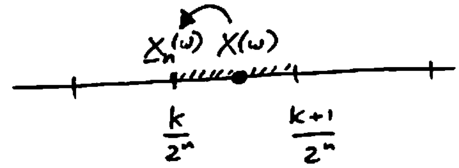
$$= \mathbb{E}[X]\cdot\mathbb{E}[Y]. \qquad \square$$

<u>Remark</u> (<u>Variance</u>) Its definition $\left(\mathrm{Var}(X) = \mathbb{E}[X - \mathbb{E}(X)]^2\right)$ and properties are ~~to~~ the same for general r.v's as for discrete.

Expectation of continuous r.v.'s.

THM 3.11 Let X be a continuous r.v. with density f. Then
$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$
The expectation exists iff the integral converges absolutely.

⌈ · (identity):



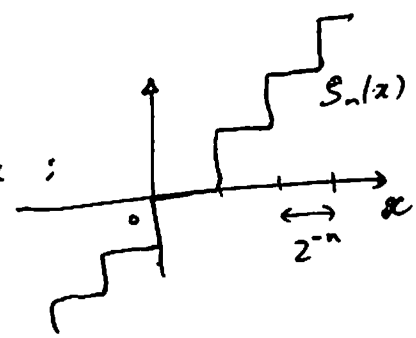
$$\mathbb{E}(X) = \lim_{n\to\infty} \mathbb{E}X_n \qquad (\text{def of } \mathbb{E} \text{ for general})$$

$$= \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} \frac{k}{2^n} \underbrace{\mathbb{P}\{X_n = \tfrac{k}{2^n}\}} \qquad (\text{def of } \mathbb{E} \text{ for discrete})$$

$$\mathbb{P}\{\tfrac{k}{2^n} < X \le \tfrac{k+1}{2^n}\} = \int_{k/2^n}^{(k+1)/2^n} f(x)\,dx \qquad (\text{def of density})$$

$$= \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} \frac{k}{2^n} \int_{k/2^n}^{(k+1)/2^n} f(x)\,dx$$

$\frac{k}{2^n} = s_n(x)$, the diadyc approximation of x:



$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} s_n(x) f(x)\,dx$$

- To complete the proof, need to replace $s_n(x)$ by x.

$$|s_n(x) - x| \le 2^{-n} \quad \forall x.$$

$$\Rightarrow \left| \int_{-\infty}^{\infty} s_n(x) f(x)\,dx - \int_{-\infty}^{\infty} x f(x)\,dx \right| \overset{\Delta}{\le} \int_{-\infty}^{\infty} \underbrace{|s_n(x) - x|}_{\le 2^{-n}} f(x)\,dx$$

$$\le 2^{-n} \underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{=1} = 2^{-n} \to 0.$$

Thus the limit $= \int_{-\infty}^{\infty} x f(x)\,dx$. □

Similarly,

Prop 3.11 (Expectation of a function of a r.v.)

Let X be a continuous r.v. with density f, and let $g: \mathbb{R} \to \mathbb{R}$ be (Borel measurable) function. Then

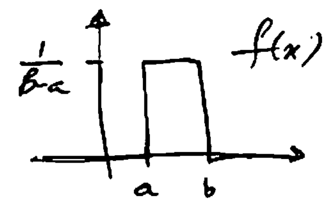
$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx .$$

3.5 SPECIAL CONTINUOUS DISTR'S.

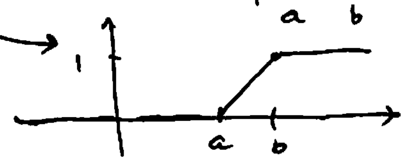
Uniform distn

Def $X \sim \mathrm{Unif}[a,b]$, $a<b$, if X has density

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{———} \end{cases}$$



CDF: $F(x) = \int_{-\infty}^{x} f(x)\, dx$



Prop $\mathbb{E}[X] = \frac{a+b}{2}$, $\quad \mathrm{Var}(X) = \frac{(b-a)^2}{12}$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \frac{1}{b-a} \int_a^b x\, dx = \frac{1}{b-a} \cdot \left[\frac{b^2}{2} - \frac{a^2}{2}\right] = \frac{a+b}{2}.$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \frac{1}{b-a} \int_a^b x^2\, dx = \frac{1}{b-a}\left(\frac{b^3}{3} - \frac{a^3}{3}\right) = \frac{1}{3}(b^2 + ab + b^2)$$

$$\Rightarrow \mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{(b-a)^2}{12}.$$

Ex When one files a tax return, one has to add some n dollar amounts. If one rounds off each amount to the nearest whole dollar, what overall error should one expect?

Probabilistic model. D_i = exact amounts, (e.g. \$178.43)

Z_i = roundoff errors $\sim$ Unif $[-\frac{1}{2}, \frac{1}{2}]$ independent (−\$0.43)

- Each amount is rounded off to $X_i = D_i + Z_i$ (\$178)
- Total is $\sum_{i=1}^{n} X_i$

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i) \quad \text{(by independence)}$$

$$= \sum_{i=1}^{n} \mathrm{Var}(Z_i) = \frac{n}{12} \quad \left(\text{since } \mathrm{Var}(Z_i) = \frac{1}{12}, \text{ see Uniform distribution}\right)$$

St. dev. of $\sum_{i=1}^{n} X_i$ is $\sqrt{\frac{n}{12}}$ = Ans

$\sqrt{\frac{n}{12}} \ll \frac{n}{2}$ (worst-case error). E.g. $n = 50$, $\sqrt{\frac{n}{12}} = \$2.0$ (while $\frac{n}{2} = \$25$)

- Application: quantization (probabilistic models)

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★ **Example 10.14 (Investment)** Dr. Caprio has invested money in three uncorrelated assets; 25% in the first one, 43% in the second one, and 32% in the third one. The means of the annual rate of returns for these assets, respectively, are 10%, 15%, and 13%. Their standard deviations are 8%, 12%, and 10%, respectively. Find the mean and standard deviation of the annual rate of return for Dr. Caprio's total investment.

Solution: Let X be the annual rate of return for Dr. Caprio's total investment. Let X_1, X_2, and X_3 be the annual rate of returns for the first, second, and third assets, respectively. By Example 4.25,

$$X = 0.25X_1 + 0.43X_2 + 0.32X_3.$$

Thus

$$E(X) = 0.25E(X_1) + 0.43E(X_2) + 0.32E(X_3)$$
$$= (0.25)(0.10) + (0.43)(0.15) + (0.32)(0.13) = 0.1311.$$

Since the assets are uncorrelated, by (10.11),

$$\mathrm{Var}(X) = (0.25)^2\,\mathrm{Var}(X_1) + (0.43)^2\,\mathrm{Var}(X_2) + (0.32)^2\,\mathrm{Var}(X_3)$$
$$= (0.25)^2(0.08)^2 + (0.43)^2(0.12)^2 + (0.32)^2(0.10)^2 = 0.004087.$$

Therefore, $\sigma_X = \sqrt{0.004087} = 0.064$. Hence Dr. Caprio should expect an annual rate of return of 13.11% with standard deviation 6.4%. Note that Dr. Caprio has reduced the standard deviation of his investments considerably by diversifying his investment; that is, by not putting all of his eggs in one basket.

eggs