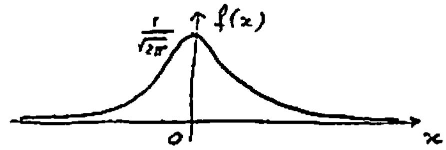
(02/02)

3.5. Standard Normal distribution

Def A rv X has the standard normal distribution if the pdf of X is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad -\infty < x < \infty.$$



Notation:

$$X \sim N(0,1).$$

- Gauss used normal distr to model his observations in astronomy so "normal" is often called "Gaussian"

- Why $\frac{1}{\sqrt{2\pi}}$?

Prop $f(x)$ above is indeed a pdf, i.e.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Proof Need to show:

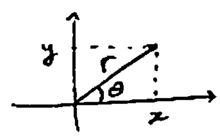
$$I := \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

Trick: $I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right) = \int_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\,dx\,dy$ (by Fubini Thm).

Pass to polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$dx\,dy = r\,dr\,d\theta.$$

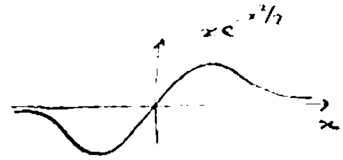


$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\,r\,dr\,d\theta = 2\pi\int_0^{\infty} r e^{-r^2/2}\,dr \underset{\substack{u = r^2/2 \\ du = r\,dr}}{=\!=\!=} 2\pi\int_0^{\infty} e^{-u}\,du = 2\pi.$$

Hence $I = \sqrt{2\pi}$. QED.

Prop For $X \sim N(0,1)$, $\boxed{E[X] = 0 \text{ and } Var(X) = 1}$ (Thus $\sigma_X = 1$)

Proof $E[X] = \frac{1}{\sqrt{2\pi}} \int_0^\infty \underbrace{x e^{-x^2/2}}_{\text{odd function}} dx = 0$



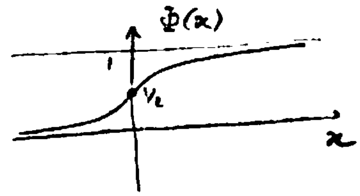
$Var(X) = E[X^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx$ $=$

(By parts $u = x$, $dv = x e^{-x^2/2} dx \Rightarrow v = \int x e^{-x^2/2} dx = -e^{-x^2/2}$)

$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{-x e^{-x^2/2} \Big|_{-\infty}^{\infty}}_{=0} + \int_{-\infty}^{\infty} e^{-x^2/2} dx \right] = 1$ by Prop. p. 48. QED.

• **Cdf of $N(0,1)$** is denoted $\Phi(a)$, recall $\Phi(a) = P\{X \le a\}$

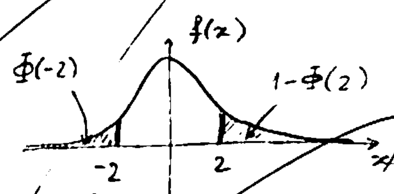
$$\boxed{\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx, \quad x \in \mathbb{R}}$$



$\Phi(x)$ can not be expressed in terms of elementary functions like $\sin x$, e^x, etc.
It is a "special function", Tabulated on p 201 of Ross

On a computer, can be computed as $\Phi(x) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)$ ← "error function", another special function

Ex Let $X \sim N(0,1)$. Compute $P\{|X| \le 2\}$



$P\{-2 \le X \le 2\} = \Phi(2) - \Phi(-2)$.

By the symmetry of the pdf $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ (even function), we have

$\Phi(-2) = 1 - \Phi(2)$

$\Rightarrow P\{|X| \le 2\} = 2\Phi(2) - 1 = 0.954$

• $\Rightarrow$ fast tail decay of $f(x)$ Ex $P\{|X| \le 3\} = 0.997$

• **Remark**: By the same symmetry reasoning

$\boxed{\Phi(-x) = 1 - \Phi(x)}$ for all x (*)

Ex Let $X \sim N(0,1)$ and $Y = \mu + \sigma X$ $\quad (\mu \in \mathbb{R}, \sigma > 0)$

Find the pdf, mean, variance of Y.

CDF: $F_Y(x) = \mathbb{P}\{\mu + \sigma X \le x\} = \mathbb{P}\{X \le \frac{x-\mu}{\sigma}\} = F_X\left(\frac{x-\mu}{\sigma}\right)$.

PDF: $f_Y(x) = \frac{d}{dx} F_Y(x) = \frac{1}{\sigma} f_X\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/\sigma}$.

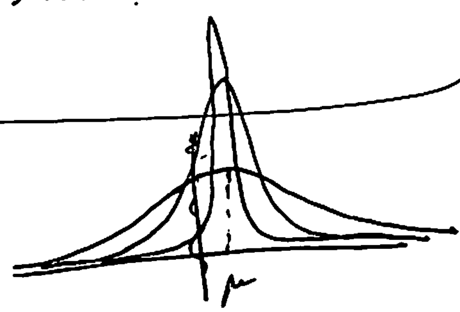
Mean: $\mathbb{E}Y = \mathbb{E}[\mu + \sigma X] = \mu$

Var: $Var(Y) = \sigma^2 Var(X) = \sigma^2$.

Def (General normal distr) $X \sim N(\mu, \sigma^2)$ if X has density

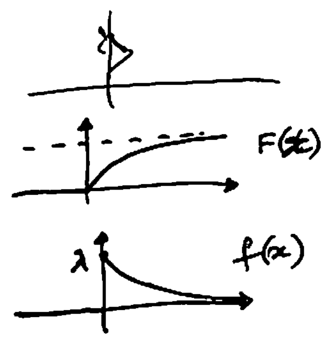
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/\sigma^2}.$$

X has mean μ and variance σ^2.



Exponential distribution

Def $X \sim Exp(\lambda)$ if $\mathbb{P}\{X > t\} = e^{-\lambda t}$, $t \ge 0$.
"rate"



$\Rightarrow$ CDF: $F(t) = 1 - e^{-\lambda t}$

PDF: $f(x) = F'(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$
$$= \int_0^{\infty} \lambda x e^{-\lambda x}\,dx = \frac{1}{\lambda}\int_0^{\infty} y e^{-y}\,dy = \ldots \text{(by parts)} = \frac{1}{\lambda}.$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \ldots = \frac{2}{\lambda^2}. \qquad \Rightarrow Var(x) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.$$

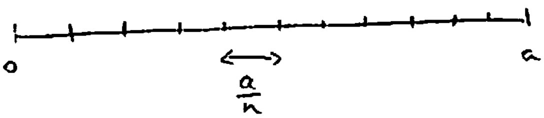
WAITING TIME

- Exponential distr is a typical model for waiting time (continuous) until some event occurs.

WHY?

Ex X = time of the first call at a police station (say, after midnight).
Let's find the distribution of X.

$P\{X \le a\} = ?$ Divide $[0,a]$ into n intervals of length $\frac{a}{n}$ (n large; e.g. every interval is a second)



$P\{\text{call during a given interval}\} = \lambda \cdot \frac{a}{n}$, for some coeff. λ (the prob. is proportional to the length of the interval)

$\Rightarrow P\{X > a\} = P\{\text{no call in } [0,a]\} = P\{\text{no call during each interval}\}$

$= \left(1 - \frac{\lambda a}{n}\right)^n$ (by independence)

$\to e^{-\lambda a}$ as $n \to \infty$

Hence $\boxed{X \sim Exp(\lambda)}$

- **Prop** (Memoryless Property) Let $X \sim Exp(\lambda)$. Then

$$P\{X > t+s \mid X > t\} = P\{X > s\} \quad \text{for } s,t > 0$$

$P\{\text{need to wait} > s \text{ more minutes after having waited } t \text{ min}\}$ $P\{\text{need to wait} > s \text{ minutes}\}$.

Proof: L.H.S $= \dfrac{P\{X > t+s\}}{P\{X > t\}} = \dfrac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = $ RHS. QED.

THM Exponential distribution is the only continuous memoryless distribution

Applications. All of these have exponential distribution.
- time until next e-mail message arrives
- time until next customer enters the store
- lifetime of equipment (e.g. a cellphone) if there is no aging; can only die due to an accident
- time between successive independent events (accidents, e-mails, payments, etc)

Exponential distr. is a "continuous version" of geometric distr:

Prop $\boxed{\text{Let } X \sim \mathrm{Exp}(\lambda). \text{ Then } \lceil X \rceil \sim \mathrm{Geom}(p), \text{ where } p = 1 - e^{-\lambda}.}$

$$\mathbb{P}\{\lceil X \rceil = k\} = \mathbb{P}\{k-1 < X \le k\} = \mathbb{P}\{X > k-1\} - \mathbb{P}\{X > k\}$$
$$= e^{-\lambda(k-1)} - e^{-\lambda k} = e^{-\lambda(k-1)}\underbrace{(1-e^{-\lambda})}_{= p} = (1-p)^{k-1} \cdot p.$$

Ex A small store is visited by 3 customer per hour on average. What is the prob that the first two customers will enter within 5 min from each other?

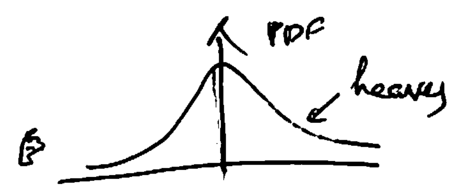
Memoryless ⇒ start the clock after 1^{st} customer enter.

X = time until next customer $\sim \mathrm{Exp}(3)$.

$$\mathbb{P}\{X < \underbrace{\tfrac{1}{12}}_{= 5\text{ min}}\} = 1 - e^{-3 \cdot \frac{1}{12}} = 0.22.$$

Cauchy distribution

$X \sim$ Cauchy if X has density
$$f(x) = \frac{1}{\pi(1+x^2)}.$$



Ex (Check $\int_{-\infty}^{\infty} f\,dx = 1$).

X does NOT have expectation:
$$\mathbb{E}|X| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|x|}{\pi(1+x^2)}\,dx = +\infty.$$
$\sim \frac{1}{x}$ as $x \to \infty$

Example: ~~Intensity~~ the Intensity of light is Cauchy