Feb 7

## 3.6. Joint distributions.

Let $X, Y$ be r.v's.

**Def** Joint CDF is
$$F(x,y) := P\{X \le x, Y \le y\}.$$

Joint PMF (if $X, Y$ discrete) is
$$p(x,y) = P\{X = x, Y = y\}.$$

Joint PDF $f(x,y) \ge 0$ exists if
$$F(\alpha,\beta) = \int_{-\infty}^{\beta} \int_{-\infty}^{\alpha} f(x,y)\,dx\,dy$$

**Prop** (Properties of joint density)

(i) $P\{a < X \le b,\ c < Y \le d\} = \int_c^d \int_a^b f(x,y)\,dx\,dy$ (check!)

(ii) More generally, $\forall$ Borel set $A \subset \mathbb{R}^2$,
$$P\{(X,Y) \in A\} = \iint_A f(x,y)\,dx\,dy$$

~~(iii) $f(x,y) \ge 0 \ \forall x, y$~~

(iii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$. ( ~~$\cancel{p\{(x,y)}$~~ (ii) with $A = \mathbb{R}^2$ ).

(iv) $f(x,y) = \dfrac{\partial^2}{\partial x\,\partial y} F(x,y)$

(v) $E\,g(X,Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)\,f(x,y)\,dx\,dy$.

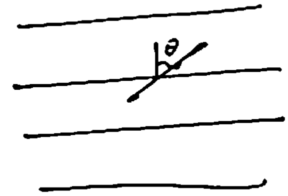
(Independence) • If $X, Y$ discrete, indep $\Rightarrow$ $\boxed{p(x,y) = p(x)\,p(y)}$

**Prop. 3.32** Suppose $X, Y$ have a ~~continuous~~ joint density $f(x,y)$.
Then $X, Y$ are independent $\Leftrightarrow$
$$f(x,y) = f(x)\,f(y) \quad \forall x, y.$$

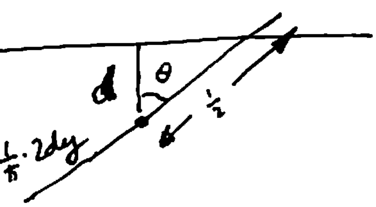
~~Similarly, for discr~~

($\Rightarrow$) By def. of $F(x,y)$ & independence,
$$F_{XY}(x,y) = F_X(x)\,F_Y(y).$$
$$\Rightarrow f_{XY}(x,y) = \frac{\partial^2}{\partial x\,\partial y} F_{XY}(x,y) = \frac{\partial}{\partial x} F_X(x) \cdot \frac{\partial}{\partial y} F_Y(y) = f_X(x) \cdot f_Y(y).$$

**Ex** (Buffon Problem) Drop a needle of length 1 onto a lined paper with distance 1 between lines. Compute the probability of intersection.




- ~~the distance~~ $\theta :=$ angle with the vertical line;
  $d :=$ distance from the center of needle to the nearest line.
- Model: $\theta \sim \text{Unif}[-\frac{\pi}{2}, \frac{\pi}{2}]$; $d \sim \text{Unif}[0, \frac{1}{2}]$, independent. (Why?)




- Intersection iff $d \le \frac{1}{2}\cos\theta$

$$\Rightarrow P\{\text{intersection}\} = P\{d \le \tfrac{1}{2}\cos\theta\} = \iint_{\{y \le \frac{1}{2}\cos x\}} \underbrace{f_\theta(x)}_{\frac{1}{\pi}} \underbrace{f_d(y)}_{2}\, dx\, dy$$

$$= \int_{-\pi/2}^{\pi/2} dx \int_0^{\frac{1}{2}\cos x} \frac{1}{\pi}\cdot 2\, dy = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{2}\cos x\, dx = \frac{1}{\pi}\sin x \Big|_{-\pi/2}^{\pi/2} = \boxed{\frac{2}{\pi}}.$$

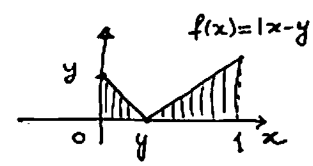
Implications: compute $\pi$ by repeating.

**Remark** (Multivariate) — straightforward extension to $X_1, X_2, \dots, X_n$:
$F(x_1, \dots, x_n)$; $f(x_1, \dots, x_n)$.

**Ex** Two people agreed to meet btw 12:00 & 1:00 pm. They arrive @ random times, uniformly over this interval, independently. What is the expected waiting time?

~~Soln~~ $X, Y \sim \text{Unif}[0,1]$ indep $E[|X-Y|] = ?$

$$E[|X-Y|] = \int_0^1 \underbrace{\int_0^1 |x-y|\, dx}_{\frac{y^2}{2} + \frac{(1-y)^2}{2}}\, dy$$




$= \dots = \frac{1}{3} = 20$ min.

## 3.6 Marginal distributions

**Def** Let $X, Y$ be a pair of r.v's.
The individual distributions of $X, Y$ are called
marginal distr's. ~~in this ca~~

**Prop** (Marginal densities) Let $(X,Y)$ have a joint density $f(x,y)$.
Then the densities of $X$, $Y$ can be recovered as

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \qquad (\text{"integrate out" } Y)$$

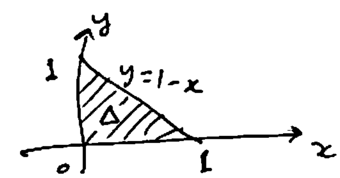
$$f_Y(x) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$\mathbb{P}\{X \in A\} = \mathbb{P}\{X \in A,\ Y \in (-\infty,\infty)\} = \int_A \underbrace{\int_{-\infty}^{\infty} f(x,y)\,dy}\,dx \qquad \text{(def of joint PDF)}$$

$$\Rightarrow f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \qquad \text{(def of density)}$$

**Ex** (from 425 p.67): Let $(X,Y)$ have joint PDF

$$f(x,y) = \begin{cases} kxy & \text{if } x \geq 0,\ y \geq 0,\ x+y \leq 1 \\ 0, & \text{———} \end{cases}$$



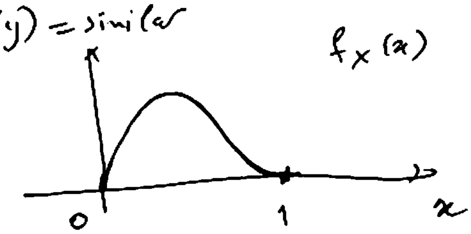

Compute $k$ & marginal PDF's of $X, Y$.

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \qquad \iint_\Delta kxy\,dx\,dy = 1 \Rightarrow k = \dfrac{1}{\iint_\Delta xy\,dx\,dy} = \cdots = 24.$

(k):

Marginals: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{1-x} 24xy\,dy = \cdots = \begin{cases} 12x(1-x)^2, & 0 \leq x < 1 \\ 0, & \text{———} \end{cases}$

$f_Y(y) = \text{similar}$




**Remark** For discrete distr's, similarly: $P_X(x) = \sum_y P_{XY}(x,y)$;
$P_Y(y) = \sum_x P_{XY}(x,y)$.

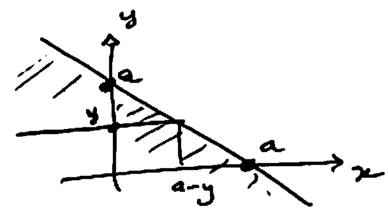
## 3.7.1 Sums of independent r.v's.

Let X, Y be indep. r.v's with known distr's. Distr of X+Y ?

Continuous case: compute $f_{X+Y}$ from $f_X$, $f_Y$ ?

• CDF:

$$F_{X+Y}(a) = \mathbb{P}\{X+Y \le a\} = \iint_{x+y\le a} f(x,y)\,dx\,dy$$




$$= \int_{-\infty}^{\infty}\int_{-\infty}^{a-y} f(x,y)\,dx\,dy = \left| x = \overset{\text{new var.}}{z} - y \right|$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{a} f(z-y,y)\,dz\,dy \overset{\text{Fubini}}{=\!=} \int_{-\infty}^{a} \underbrace{\left[\int_{-\infty}^{\infty} f(z-y,y)\,dy\right]}_{= f_{X+Y}(z) \text{ by def. of density}} dz$$

~~⇒ indep~~ Moreover, if X, Y indep ⇒

$$f(z-y,y) = f_X(z-y)\, f_Y(y).$$

We proved:

**Prop** (3.4.4) Let X, Y be indep. r.v's ~~Then the~~ with densities $f_X$, $f_Y$. Then X+Y has density

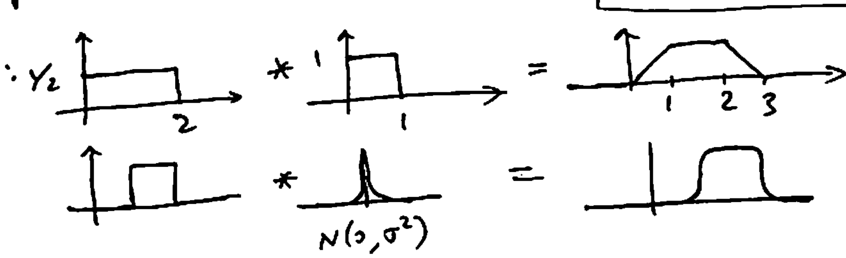
$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(z-y)\, f_Y(y)\,dy = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\,dx$$

**Def** Convolution of $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$ is the function $f*g:\mathbb{R}\to\mathbb{R}$ defined as

$$(f*g)(z) = \int_{-\infty}^{\infty} f(z-y)\, g(y)\,dy = \int_{-\infty}^{\infty} f(x)\, g(z-x)\,dx.$$

We proved: if X, Y indep then $\boxed{f_{X+Y} = f_X * f_Y.}$

Ex:




(Smoothing).

Sums of indep. r.v's with special distributions.

~~Normal~~ Before: $\Sigma$ Ber = Binom; $\Sigma$ Binom = Binom; $\Sigma$ Poisson = Poisson

Normal.

Prop let $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ be indep. Then

$$X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2).$$

First, for $\mu_1=\mu_2=0$, $\sigma_1=\sigma_2=1$ : $X, Y \sim N(0,1)$.

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f(z-y) f(y)\, dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(z-y)^2/2} e^{-y^2/2}\, dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\underbrace{(y^2 - zy + z^2/2)}_{= \left[\left(y-\frac{z}{2}\right)^2 + \frac{z^2}{4}\right]}}\, dy = e^{-z^2/4}\cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(y-\frac{z}{2}\right)^2} dy$$

$\left(y-\frac{z}{2}\right)^2 = \frac{x^2}{2}$, $y - \frac{z}{2} = \frac{x}{\sqrt{2}}$

$$e^{-z^2/4}\cdot \frac{1}{2\sqrt{2}\pi} \underbrace{\int_{-\infty}^{\infty} e^{-x^2/2}\, dx}_{=\sqrt{2\pi}} = \frac{1}{2\sqrt{\pi}} e^{-z^2/4} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-z^2/2\sigma^2} \text{ for } \sigma=\sqrt{2}$$

$\Rightarrow$ $X+Y \sim N(0,2)$.

~~(2) Now for general $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$.~~

~~$X = \mu_1 + \sigma_1 \bar{X}$, $Y = \mu_2 + \sigma_2 \bar{Y}$, $\bar{X}, \bar{Y} \sim N(0,1)$~~

~~$\Rightarrow X+Y = (\mu_1+\mu_2) + \sigma_1$~~

For general $X, Y$ : check!

~~More general:~~ By induction $\Downarrow$

Thm let $X_i \sim N(\mu_i, \sigma_i^2)$ be indep. Then

$$\sum_{i=1}^n X_i \sim N(\mu, \sigma^2), \quad \text{where } \mu = \sum_{i=1}^n \mu_i, \quad \sigma^2 = \sum_{i=1}^n \sigma_i^2.$$