3.6. Joint distributions.

Let $X, Y$ be r.v.'s.

**Def.** Joint CDF is

$$F(x, y) = P\{X \leq x, Y \leq y\}.$$

Joint PMF (if $X, Y$ discrete) is

$$p(x, y) = P\{X = x, Y = y\}.$$

Joint PDF $f(x, y)$ exists if

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x, y) \, dx \, dy.$$

**Prop. (Properties of joint density)**

(i) \( P\{a < X \leq b, c < Y \leq d\} = \int_{a}^{b} \int_{c}^{d} f(x, y) \, dx \, dy \) (Check!)

(ii) More generally, for a box set $A \subseteq \mathbb{R}^2$,

$$P\{(X, Y) \in A\} = \iint_{A} f(x, y) \, dx \, dy.$$

(iii) $f(x, y) \geq 0$ for $x, y$.

(iv) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1$.

(v) $f(x, y) = \frac{\partial}{\partial x} F(x, y)$

(vi) $E g(X, Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) \, dx \, dy.$

(Independence) If $X, Y$ discrete, independent $\Rightarrow P(x, y) = p(x) p(y)$.

**Prop. 3.32** Suppose $X, Y$ have a continuous joint density $f(x, y)$. Then $X, Y$ are independent $\iff$

$$f(x, y) = f(x) f(y) \quad \forall x, y.$$

By def. of $F(x, y)$ & independence,

$$F_{x,y}(x, y) = F_x(x) F_y(y).$$

$$f_{x,y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{x,y}(x, y) = \frac{\partial}{\partial x} F_x(x) \cdot \frac{\partial}{\partial y} F_y(y) = f_x(x) \cdot f_y(y).$$
Ex. (Buffon Problem) Drop a needle of length $L$ onto a grid paper with distance $t$ between lines. Compute the probability of intersection.

- $t = \text{distance}$ to nearest line.
- $d = \text{vertical component}$ of distance from center of needle.
- $d = \text{distance} \sqrt{1 - \cos^2 \theta} = \text{distance} \sin \theta$
- $\theta = \frac{d}{\text{distance}}$
- $P(\text{intersection}) \propto \frac{d}{\text{distance}}$
- $P(\theta = \frac{d}{\text{distance}} = \frac{L}{2}) = \frac{L}{2\text{distance}}$
3.6 Marginal Distributions

**Def.** Let \( X, Y \) be a pair of r.v.'s. The individual distributions of \( X, Y \) are called **marginal distributions**.

**Prop. (Marginal Densities)** Let \((X, Y)\) have a joint density \( f(x, y) \). Then the densities of \( X, Y \) can be recovered as:

\[
\begin{align*}
    f_X(x) &= \int_{-\infty}^{\infty} f(x, y) \, dy \\
    f_Y(y) &= \int_{-\infty}^{\infty} f(x, y) \, dx
\end{align*}
\]

("integrate out" \( Y \))

\[
P\{X \in A\} = P\{X \in A, Y \in (-\infty, \infty)\} = \int_{A} \int_{-\infty}^{\infty} f(x, y) \, dy \, dx \quad \text{(def of joint PDF)}
\]

\[
\Rightarrow f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy \quad \text{(def of density)}
\]

**Ex. (from 12.5 p.67):** Let \((X, Y)\) have joint PDF

\[
f(x, y) = \begin{cases} 
    kxy & \text{if } x \geq 0, y \geq 0, x+y \leq 1 \\
    0 & \text{otherwise}
\end{cases}
\]

Compute \( k \) & marginal PDF's of \( X, Y \).

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dy \, dx = 1 \\
\int_{0}^{1} \int_{0}^{1-x} kxy \, dx \, dy = 1 \Rightarrow k = \frac{1}{\int_{0}^{1} \int_{0}^{1-x} xy \, dy \, dx} = \cdots = \frac{1}{24}
\]

**K:**

\[
\begin{align*}
    f_X(x) &= \int_{-\infty}^{\infty} f(x, y) \, dy = \int_{-\infty}^{1-x} kxy \, dy = \cdots = \begin{cases} 
    12x(1-x)^2, & 0 \leq x \leq 1 \\
    0, & \text{otherwise}
\end{cases} \\
    f_Y(y) &= \int_{-\infty}^{\infty} f(x, y) \, dx = \cdots = \begin{cases} 
    12y^2, & 0 \leq y \leq 1 \\
    0, & \text{otherwise}
\end{cases}
\end{align*}
\]

**Remark.** For discrete distr's, similarly:

\[
P_X(x) = \sum_{y} P_{X,Y}(x, y) \\
P_Y(y) = \sum_{x} P_{X,Y}(x, y)
\]

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3.7.1 Sums of independent r.v.'s.

Let $X, Y$ be indep. r.v.'s with known dist's. Dist of $X+Y$?

Continuous case: compute $f_{X+Y}$ from $f_X$, $f_Y$?

CDF:

$$F_{X+Y}(a) = P\{X+Y \leq a\} = \iiint f_{X+Y}(x,y) \, dx \, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{a-y} f(x,y) \, dx \, dy$$

$$= \int_{-\infty}^{a} \int_{-\infty}^{\infty} f(2-y,y) \, dy \, dx$$

Fubini's theorem implies:

$$f_{X+Y}(a) \text{ by def. of density.}$$

Moreover, if $X, Y$ indep.:

$$f(2-y,y) = f_X(2-y) f_Y(y).$$

We proved:

**Prop.** (3.4.7) Let $X, Y$ be indep. r.v.'s. Then the r.v. $X+Y$ has density

$$f_{X+Y}(z) = \int_{-\infty}^{z} f_X(z-y) f_Y(y) \, dy = \int_{-\infty}^{\infty} f_X(z-x) f_Y(z-x) \, dx$$

**Def.** Convolution of $f, g : R \rightarrow R$ is the function $f \ast g : R \rightarrow R$ defined as

$$(f \ast g)(z) = \int_{-\infty}^{\infty} f(z-y) g(y) \, dy = \int_{-\infty}^{\infty} f(x) g(z-x) \, dx$$

We proved: If $X, Y$ indep. then $f_{X+Y} = f_X \ast f_Y$.

$E_X : Y \xrightarrow{\ast} f_X \ast f_Y = f_X \ast f_Y$ (Smoothing).

\[\begin{array}{c@{\ast}c@{\ast}c}
\text{Ex} & \text{Y} & \text{f}_X \\
\hline
2 & 1 & 1 \\
\hline
1 & \text{Normal distr. (N)} & \text{f}_X \ast f_Y \\
\end{array}\]
Sums of independent r.v.s with special distributions.

\textbf{Normal.}

Let \( X \sim N(\mu_1, \sigma_1^2) \), \( Y \sim N(\mu_2, \sigma_2^2) \) be independent. Then \( X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2) \).

\[ f_{X+Y}(z) = \int_{-\infty}^{\infty} f_{X}(z-y) f_{Y}(y) \, dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(z-y)^2/2} e^{-y^2/2} \, dy = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(z-y^2/2 + z/2)} \, dy = e^{-z^2/4} \cdot \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(y^2/2 + z/4)} \, dy = e^{-\frac{z^2}{8}} \cdot \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2\sigma^2}} \text{ for } \sigma = \sqrt{2}
\]

\[ \Rightarrow X + Y \sim N(0, 2).
\]

(2) Now for general \( X \sim N(\mu_1, \sigma_1^2) \), \( Y \sim N(\mu_2, \sigma_2^2) \).

\[ X = \mu_1 + \tilde{X}, \quad Y = \mu_2 + \tilde{Y}, \quad X, Y \sim N(0, 1)
\]

\[ \Rightarrow X + Y = (\mu_1 + \mu_2) + 0,
\]

For general \( X, Y \) : check.

More general: By induction \( \nabla \)

Thus, let \( X_i \sim N(\mu_i, \sigma_i^2) \) be independent. Then
\[ \sum_{i=1}^{n} X_i \sim N(\mu, \sigma^2), \quad \text{where } \mu = \sum_{i=1}^{n} \mu_i, \quad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2.
\]