Feb 9

Σ Exponential

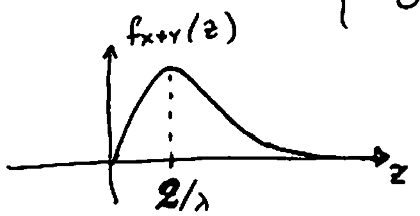
Ex: Service time @ airport check-in desk $\sim Exp(\lambda)$.
Distribution of service time for a customer who is 2^{nd} in line?

$X, Y \sim Exp(\lambda)$ indep. $X+Y \sim$?

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f(z-y) f(y)\,dy = \int_{0}^{z} \lambda e^{-\lambda(z-y)} \cdot \lambda e^{-\lambda y}\,dy \;=$$

(recall pdf of $Exp(\lambda)$: $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$)

$$= \lambda^2 e^{-\lambda z} \int_0^z dy = \begin{cases} \lambda^2 z e^{-\lambda z}, & z \geq 0 \\ 0, & z < 0 \end{cases}$$



Continue adding new ~~Exp~~ $Exp(\lambda)$ r.v's $\Rightarrow$

Ex The pdf of $X_1 + X_2 + \dots + X_n$, for $X_i \sim Exp(\lambda)$ indep, is proportional to $(\lambda z)^{n-1} \cdot \lambda e^{-\lambda z}$. (Check!)

~~Coefficient~~? $\Rightarrow$ $f_n(z) = \frac{1}{C} (\lambda z)^{n-1} (\lambda e^{-\lambda z})$, $z \geq 0$

$C = ?$ $\int_0^{\infty} f_n(z)\,dz = 1 \Rightarrow C = \int_0^{\infty} (\underbrace{\lambda z}_{= t})^{n-1} (\lambda e^{-\lambda z})\,dz = \int_0^{\infty} t^{n-1} e^{-t}\,dt = (n-1)!$

$$\Rightarrow \boxed{f_n(z) = \frac{1}{(n-1)!} (\lambda z)^{n-1} \lambda e^{-\lambda z}, \quad z > 0}$$

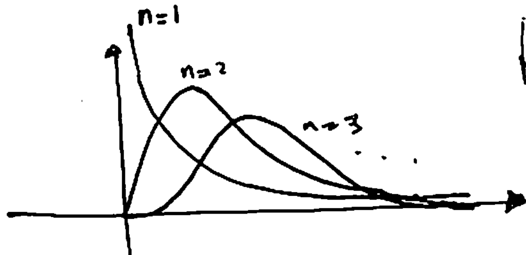
Rem Extension of this identity from $n \in \mathbb{N}$ to $n \in \mathbb{R}$ gives the def. of Gamma function

$$\boxed{\int_0^{\infty} t^{n-1} e^{-t}\,dt =: \Gamma(n)}, \quad \Gamma(n) = (n-1)!, \; n \in \mathbb{N}.$$

Def (Gamma distribution)

$X \sim \text{Gamma}(n, \lambda)$, $n, \lambda > 0$, if X has pdf

$$f(x) = \begin{cases} \frac{1}{\Gamma(n)} (\lambda x)^{n-1} \cdot \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$



$\text{Exp}(1, \lambda) = \text{Gamma}(1, \lambda)$

Prop 1. If $X_1, \dots, X_n \sim \text{Exp}(\lambda)$ indep $\Rightarrow$ $X_1 + \dots + X_n \sim \text{Gamma}(n, \lambda)$

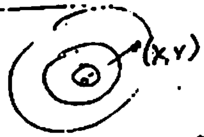
2. If $X_1, \dots, X_n \sim \text{Gamma}(n_i, \lambda)$ indep $\Rightarrow$ $X_1 + \dots + X_n \sim \text{Gamma}(\Sigma n_i, \lambda)$.

1: already proved.

2 (for $n_i \in \mathbb{N}$): represent each X_i as a sum of n_i exponentials. use first part.

Prop $X \sim \text{Gamma}(n, \lambda) \Rightarrow \mathbb{E}[X] = \frac{n}{\lambda}$, $\text{Var}(X) = \frac{n}{\lambda^2}$

(For $n \in \mathbb{N}$): Follows by Prop (part 1) & $\mathbb{E}[X_i] = \frac{1}{\lambda}$, $\text{Var}(X_i) = \frac{1}{\lambda^2}$

Ex Shooting at Target  The shot (X,Y): $X, Y \sim N(0,1)$ indep.

Distr. of $R^2 = X^2 + Y^2$?

Distr. of X^2 ?

$$F_{X^2}(a) = P\{X^2 \le a\} = P\{-\sqrt{a} \le X \le \sqrt{a}\} = \Phi(\sqrt{a}) - \Phi(-\sqrt{a})$$

$$= 2\Phi(\sqrt{a}) - 1$$

$$\Rightarrow \quad f_{X^2}(a) = \frac{d}{da} F_{X^2}(a) = 2\frac{d}{da}\Phi(\sqrt{a})$$

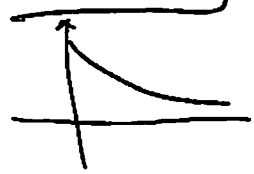
$$= 2 f_X(\sqrt{a}) \cdot \frac{1}{2\sqrt{a}} \qquad \text{where } f_X(\cdot) \text{ is the PDF of } N(0,1)$$

$$= \frac{1}{\sqrt{a}} f_X(\sqrt{a}) = \frac{1}{\sqrt{a}} \cdot \frac{1}{\sqrt{2\pi}} e^{-a/2} \qquad (\text{= Gamma ?})$$

$$= \tfrac{1}{2} e^{-a/2} (a/2)^{\frac{1}{2}-1} \cdot C$$

$$\Rightarrow \quad X^2 \sim \text{Gamma}\left(\tfrac{1}{2}, \tfrac{1}{2}\right).$$

By Prop. p. 59, $X^2 + Y^2 \sim \text{Gamma}\left(\frac{1}{2}+\frac{1}{2}, \frac{1}{2}\right) = \text{Gamma}\left(1, \frac{1}{2}\right) \sim \boxed{\text{Exp}\left(\frac{1}{2}\right)}$



$$f_{X^2+Y^2}(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & x \ge 0 \\ 0, & \end{cases}$$

Remark More generally, if $X_1, \dots, X_n \sim N(0,1)$ then

$$X_1^2 + \dots + X_n^2 \sim \text{Gamma}\left(\tfrac{n}{2}, \tfrac{1}{2}\right) =: \boxed{\chi_n^2}$$

Chi-square distr. with n degrees of freedom.

(Ex.)

Summary: $\sum$ Bernoulli = Binomial, $\sum$ Binom = Binom, $\sum$ Poisson = Poisson

$\sum$ Normal = Normal, $\sum$ Exp = Gamma, $\sum$ Gamma = Gamma, $\sum \text{Normal}^2 = \chi^2$.

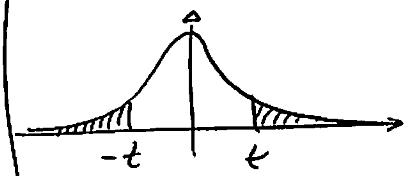
$\sum$ chi-square = chi-square.

An application:

Prop (Normal tail bound)

Let $X \sim N(0,1)$. Then $\forall t>0$,

$$\mathbb{P}\{|X| \geq t\} \leq e^{-t^2/2}.$$



let $Y \sim N(0,1)$ indep $\Rightarrow$

$$\mathbb{P}\{|X|>t\} = \sqrt{\mathbb{P}\{|X|>t, |Y|>t\}} \quad \text{(by independence, identical distr)}$$

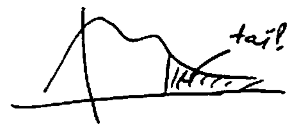
$$\leq \sqrt{\mathbb{P}\left\{\frac{X^2+Y^2}{2} \geq t^2\right\}}$$

$$= \sqrt{e^{-t^2}} \quad \left(\text{since } Z := X^2+Y^2 \sim \mathrm{Exp}(\tfrac{1}{2}) \text{ (p. 60)}, \text{ thus } \mathbb{P}\{Z>u\} \leq e^{-u/2}.\right)$$

$$= e^{-t^2/2}.$$

3.3.4 Some inequalities.

Prop (Markov's ineq.) Let X be a r.v. taking non-negative values.
Then $\forall t>0$,
$$\mathbb{P}\{X \geq t\} \leq \frac{\mathbb{E}(X)}{t}.$$



$\ulcorner$
$$\mathbb{E}(X) \geq \mathbb{E}\left[\underset{t}{\underset{\vee}{X}}\, \mathbb{1}_{\{X\geq t\}}\right] \qquad (\text{since } X \geq X\,\mathbb{1}_{\{X\geq t\}})$$
$$\geq t \cdot \mathbb{E}\left[\mathbb{1}_{\{X\geq t\}}\right] = t \cdot \mathbb{P}\{X \geq t\}.$$
$\lrcorner$.

Cor (Chebyshev's ineq.) $\forall$ r.v. X with finite variance,
$$\mathbb{P}\{|X-\mathbb{E}X| \geq t\} \leq \frac{Var(X)}{t^2}$$

$\ulcorner$ ~~Use Markov's ineq. for $|X-\mathbb{E}X|$ instead of X.~~
$$\mathbb{P}\{|X-\mathbb{E}X|>a\} = \mathbb{P}\{(X-\mathbb{E}X)^2 > t^2\}$$
$$\leq \frac{\mathbb{E}\left[(X-\mathbb{E}X)^2\right]}{t^2} \quad (\text{Markov}). \qquad \square \lrcorner$$

~~**Ex** Roll a fair die~~ ~~Toss a coin 10,000 times~~
~~$\mathbb{P}\{|X-$~~

Ex Roll a fair die $n=10{,}000$ times, $X :=$ total score.
~~$\mathbb{E}[X] = 35{,}000$, $Var(X) =$~~ $\mathbb{E}(X) = 3.5n = 35{,}000$; $Var(X) = \frac{35}{12}n$ (Check!)

Then $\mathbb{P}\{|X-35{,}000|>1{,}000\} \leq \frac{(35/12)n}{(1000)^2} \simeq 0.029.$

(Can do better!)