Feb. 14

Thm (Cauchy–Schwarz inequality) $\forall$ r.v.'s X, Y :

$$\mathbb{E}[XY] \le \sqrt{\mathbb{E}[X^2]} \cdot \sqrt{\mathbb{E}[Y^2]}$$

Equality iff $X=0$ or $Y=0$ a.s., or $X=\lambda Y$ a.s.

Let $\lambda \ge 0$ be a parameter. $\Rightarrow$

$$0 \le \mathbb{E}[(X-\lambda Y)^2]$$

$$= \mathbb{E}[X^2] - 2\lambda\, \mathbb{E}[XY] + \mathbb{E}[Y^2]$$

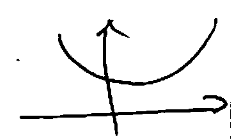
$$\Rightarrow \quad \mathbb{E}[XY] \le \frac{1}{2\lambda}\mathbb{E}(X^2) + \frac{\lambda}{2}\mathbb{E}[Y^2].$$

Optimize RHS in λ, minimum attained for $\lambda = \frac{\sqrt{\mathbb{E}[X^2]}}{\sqrt{\mathbb{E}[Y^2]}}$, ~~(Check!)~~

~~Substitute,~~ ~~and the~~ and it equals $\sqrt{\mathbb{E}[X^2]}\cdot\sqrt{\mathbb{E}[Y^2]}$. (Check!)

Thm (Jensen's inequality) Let $\phi : (a,b) \to \mathbb{R}$ be a convex function

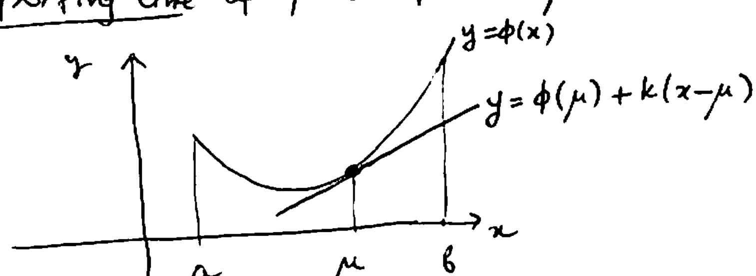
i.e. $\phi(\lambda x + (1-\lambda)y) \le \lambda\phi(x) + (1-\lambda)\phi(y) \quad \forall\, a<x,y<b, \ \forall\, 0<\lambda<1.$



Then $\forall$ r.v. X taking values in (a,b),

$$\phi\big(\mathbb{E}[X]\big) \le \mathbb{E}\big[\phi(X)\big]$$

Consider a supporting line of ϕ at point $\mu := \mathbb{E}[X]$.



Convexity $\Rightarrow$ $\phi(\mu) + k(x-\mu) \le \phi(x)$.

Substitute $X=x$ and take expectation of both sides $\Rightarrow$

$$\underbrace{\phi(\mu)}_{=\phi(\mathbb{E}[X])} + \underbrace{k(\underbrace{\mathbb{E}[X]}_{=\mu} - \mu)}_{=0} \le \mathbb{E}\,\phi(X).$$

$\square$

<u>Examples</u>. $e^{\mathbb{E}[X]} \le \mathbb{E}[e^X]$.

• $(\mathbb{E}[X])^2 \le \mathbb{E}[X^2]$ ($\Rightarrow \mathrm{Var}(X) \ge 0$.)

<u>Cor</u> (<u>Lyapunov's ineq.</u>) ~~$\forall\ 1 \le p < q$~~,

(i) $(\mathbb{E}|X|)^p \le \mathbb{E}[|X|^p]$ $\forall\ p \ge 1$

(ii) $(\mathbb{E}[|X|^p])^{1/p} \le (\mathbb{E}|X|^q)^{1/q}$ $\forall\ 1 \le p < q$

(i) Jensen for $\phi(x) := |x|^p$

(ii) Jensen for $\underset{\text{convex}}{\phi(x) = |x|^{q/p}}$, $X \mapsto |X|^p$.

~~$\phi(\mathbb{E}|X|^p$~~

$$(\mathbb{E}|X|^p)^{p/q} \le \mathbb{E}\left[|X|^{p \cdot \frac{p}{q}}\right] = \mathbb{E}|X|^q.$$

<u>Remark</u>. $(\mathbb{E}|X|^p)^{1/p} = \|X\|_{L^p}$ "<u>L^p norm</u>".

~~Lyapunov~~:

• $\mathbb{E}[XY] = \langle X, Y \rangle_{L^2}$ "<u>inner product</u>"

Thus: $\langle X, Y \rangle \le \|X\|_{L^2} \|Y\|_{L^2}$ (C-S)

$\|X\|_{L^p} \le \|X\|_{L^q}$ if $p \le q$ (Lyapunov).

Midterm 2 Review.

① (Ross 3.43) An urn contains ~~n+m balls~~ n white and m black balls. They are withdrawn one at a time, w/o replacement, until the first black ball is chosen. ~~Find the expected # of red~~ Find the expected # of white balls withdrawn.

WWWWB
$\underbrace{\text{WWWW}}_{X=4}$B

$$X = \#\text{white balls withdrawn} = \sum_{i=1}^{n} X_i, \quad \cancel{X_i = 1, \text{wh}}$$

$$X_i = \begin{cases} 1, & \text{white ball } i \text{ chosen before any black ball} \\ 0, & \text{——} \end{cases}$$

$$\mathbb{E}[X_i] = \mathbb{P}\{X_i = 1\} = \mathbb{P}\{\text{white ball } i \text{ chosen before } \forall \text{ black}\}$$

$$= \frac{1}{m+1} \quad \left(\text{each of these } m+1 \text{ balls is equally likely to be chosen earliest}\right).$$

$$\Rightarrow \mathbb{E}[X] = \frac{n}{m+1}.$$

Ex: Find Var(X).

② (Ross 3.73) Suppose we continually roll a die until the sum of all throws exceeds 100. What is the most likely value of this total? $=: X$

By conditioning on $Y :=$ the value of the sum prior to going over 100.

Value of Y	Possible values of X	Prob.
95	101	1
96	101, 102	½ each
97	101, 102, 103	⅓ each
...	— — — — —	—
100	101, 102, 103, 104, 105, 106	⅙ each

~~$\Rightarrow \mathbb{P}\{X=10$~~

$$\mathbb{P}\{X=101\} = \sum_{y=95}^{100} \mathbb{P}\{X=101 \mid Y=y\}\,\mathbb{P}\{Y=y\} = 1\cdot\mathbb{P}\{Y=95\} + \tfrac{1}{2}\cdot\mathbb{P}\{Y=96\} + \tfrac{1}{3}\cdot\mathbb{P}\{Y=97\} + \dots + \tfrac{1}{6}\,\mathbb{P}\{Y=100\}.$$

$$\mathbb{P}\{X=102\} = \quad \tfrac{1}{2}\,\mathbb{P}\{Y=96\} + \tfrac{1}{3}\,\mathbb{P}\{Y=96\} + \dots + \tfrac{1}{6}\,\mathbb{P}\{Y=100\}$$

$\Rightarrow$ SMALLER!

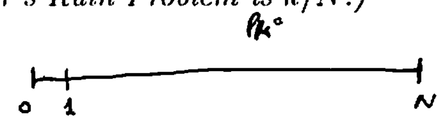
Similarly, 103, 104, 105, 106. Ans: 101.

Bonus Question *(8 extra points, no partial credit).*

Suppose that, every year, the company's fund is increased by 1 million dollars with probability 1/2 or decreased by 1 million dollars with probability 1/2. Every time the company runs out of funds (i.e. the funds become zero), the government gives it a bail-out in the amount of 1 million dollars. The initial company's fund is k million dollars. Compute the expected number of bail-outs needed to increase the company's fund to N million dollars.

(Hint: Recall the Gambler's Ruin Problem. That model is similar except the company is allowed to go bankrupt if it runs out of funds. Recall that the probability to increase the funds from k to N without going bankrupt in Gambler's Ruin Problem is k/N.)

Let $\cancel{X}$ = # of Bailouts



There are two possible scenarios:

1) The company never ~~files for bankruptcy~~ goes bankrupt before reaching N (thus $B=0$).
This happens with probability p_k.

2) The company ~~files~~ goes bankrupt before reaching N.
This happens with probability $1-p_k$.

Once bankrupt, the company is given a bailout, and with probability p_1 it reaches N ~~before the~~ without further bankrupcies. ~~next bailouts.~~
Hence the number of additional ~~bailouts is~~ bankrupcies is
$\Rightarrow$ with prob. $1-p_1$ it goes bankrupt again before reaching N.
~~Hence the~~ # of additional bankrupcies B

After this, the number of additional bailouts is, the number of hits 0 (~~starting 1~~ (before reaching N) $\Rightarrow$ it is $Y-1$, where $Y \sim Geom(p_1)$. ~~an geometric r.v. with parameter p_1.~~

~~$\Rightarrow$ mean~~ $X = \begin{cases} 0, & \text{prob. } p_k \\ Y \sim Geom(p_1), & 1-p_k \end{cases}$

↑ taking into account the first bail-out

Hence
$$E[X] = 0 \cdot p_k + E[Geom(p_1)] \cdot (1-p_k) = \frac{1-p_k}{p_1} = \frac{1-k/N}{1/N} = \boxed{N-k}.$$

Bonus Question *(8 extra points, no partial credit).*

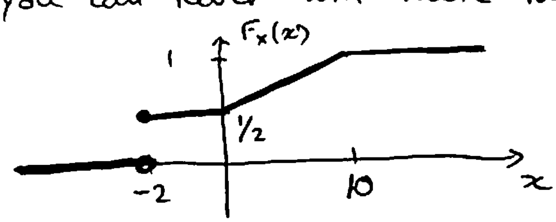
A coin is tossed. If it shows heads, you pay 2 dollars. If it shows tails, you spin a wheel which gives the amount you win distributed uniformly between 0 and 10 dollars. Your gain (or loss) is a random variable X. Find its distribution function, and compute the probability that you win at least 5 dollars.

- For $x < -2$, $F_X(x) = P\{X \le x\} = 0$
since you can never lose more than 2 dollars.

- For $-2 \le x < 0$,
$$F_X(x) = P\{X \le x\} = P\{X = -2\} = \frac{1}{2}$$
since if you lose at all, then you lose exactly 2 dollars.

- For $0 \le x < 10$,
$$F_X(x) = P\{X \le x\} = \frac{1}{2} + \frac{1}{2} \cdot \frac{x}{10}$$
because either $X = -2$ (which happens with probability $\frac{1}{2}$) or the wheel shows a number between 0 and x with probability $\frac{x}{10}$, which has to be multiplied by $\frac{1}{2}$, the probability that the wheel will be used at all.

- For $x \ge 10$,
$$F_X(x) = P\{X \le x\} = 1.$$
Since you can never win more than 10 dollars.



- $P\{X \ge 5\} = 1 - P\{X < 5\} = 1 - \lim_{x \uparrow 5} F_X(x) = 1 - F_X(5) = 1 - \frac{3}{4} = \frac{1}{4}$