03/07

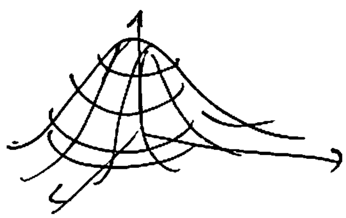
Multivariate normal distribution.

Def (Standard) $X \sim N(0, I_n)$ if $X = (X_1, \dots, X_n)$ with $X_i \sim N(0,1)$ independent.

Density? $\forall x = (x_1, \dots, x_n) \in \mathbb{R}^n$,

$$f_X(x) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} e^{-x_i^2/2} = \boxed{\frac{1}{\sqrt{2\pi}} e^{-\|x\|^2/2}}$$

where $\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$ ~~Euclid~~ (length)



Rotation invariant: $f_X(Ux) = f(x)$ $\forall$ orthogonal matrix U.

Cor (Rot. inv.) Let $X \sim N(0, I_n)$, U orthogonal matrix (fixed). Then
$$UX \sim N(0, I_n)$$

($U^T U = U U^T = I$; columns, rows orth.; $\|Ux\| = \|x\|$)

$\ulcorner$ $\forall$ Borel $B \subset \mathbb{R}^n$,

$$\mathbb{P}\{UX \in B\} = \mathbb{P}\{X \in U^{-1}B\} = \int_{U^{-1}B} f(x)\,dx \qquad \left(\text{where } f(x) = \tfrac{1}{\sqrt{2\pi}} e^{-\|x\|^2/2}\right)$$

$$= \int_B f(Uz)\,dz \qquad \left(\text{change of var. } x = Uz, \text{ Jacobian } = |U| = 1 \text{ (determinant)}\right)$$

$$= \int_B f(z)\,dz$$

$\Rightarrow f_{UX}(x) = f_X(x)$. $\lrcorner$

New proof of

Cor If $X_i \sim N(0,1)$ indep., $\sum \sigma_i^2 = 1 \Rightarrow$

$$\sum \sigma_i X_i \sim N(0,1)$$

(We proved this before!)

$\Big[\ \sum \sigma_i X_i = \langle \sigma, X \rangle$ where $\sigma = (\sigma_1, \dots, \sigma_n)$, $X = (X_1, \dots, X_n)$

$$U = \begin{bmatrix} \sigma \\ \hline U \end{bmatrix}, \quad \begin{bmatrix} \sigma \\ \hline U \end{bmatrix} \begin{bmatrix} X \end{bmatrix} = \begin{bmatrix} \langle \sigma, X \rangle \\ \vdots \\ \vdots \end{bmatrix} \sim \begin{bmatrix} N(0,1) \\ N(0,1) \\ \vdots \end{bmatrix} \quad \text{by Cor. p. 73} \ \Big] .$$

make U orthogonal.

Def (~~General~~ Normal) Let $\mu \in \mathbb{R}^n$, Σ : PSD$_{n \times n}$. $X \sim N(\mu, \Sigma)$ if $X = \mu + AZ$ where $Z \sim N(0, I_n)$ and $A = \sqrt{\Sigma}$.

$X_1, \dots, X_n$ are said to ~~have~~ be jointly Gaussian

$\Rightarrow \mathbb{E}[X] = \mu$, $\mathrm{Cov}(X) = AA^T$ (Prop. p. 72)

$= \Sigma$ (by def.).

Cor (3.53) (i) Let $\mu_1, \dots, \mu_n$, Σ : PSD.

Then $\exists$ jointly Gaussian r.v.'s $X_1, \dots, X_n$ with means $\mu_1, \dots, \mu_n$, covariance Σ.

(ii) Conversely, the means EX_i and covariances ~~matrix~~ $\mathrm{Cov}(X_i, X_j)$ uniquely determine the distr. of X_i, if (X_i) are jointly Gaussian.

~~Cor If X & Y are jointly Gauss. & uncorrelated ⇒ indep.~~

~~Remarks: (a) If we don't require (X_i) to be Gaussian, then mean & cov's do NOT determine distr!~~

(immediately follows from def.).

Cor If $X_1, \dots, X_n$ are jointly Gauss. & uncorrelated ($\mathrm{Cov}(X_i, X_j) = 0 \ \forall i,j$) then X_i's are independent.

$X := (X_1, \dots, X_n) \sim N(\mu, \Sigma)$ where $\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$ diagonal.

On the other hand, if $Z_i \sim N(0,1)$ indep, then

$Y := (\sigma_1 Z_1, \dots, \sigma_n Z_n) \sim N(\mu, \Sigma)$.

$\Rightarrow$ X, Y have same distr.

But coord's of Y are indep $\Rightarrow$ coord's of X are indep

Rem Warnings: (a) As we know, in general uncorrelated $\not\Rightarrow$ independent

(b) Even if $X_1, \dots, X_n$ have Gauss distr. (but not jointly Gauss), uncorrelated $\not\Rightarrow$ indep. (Ex. 3.41)

Prop (Density) Let $X \sim N(\mu, \Sigma)$. Then the density of X is

$$f(x) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}, \quad x \in \mathbb{R}^n.$$

By def, $X = \mu + AZ$ where $A = \sqrt{\Sigma}$ and $Z \sim N(0, I_n)$ $\Rightarrow$ $\forall$ Borel $B \subset \mathbb{R}^n$,

$$\mathbb{P}\{X \in B\} = \mathbb{P}\{\mu + AZ \in B\} = \mathbb{P}\{Z \in A^{-1}(B-\mu)\} = \int_{A^{-1}(B-\mu)} f_Z(z)\,dz \ \ominus$$

Change of var's: $x = \mu + Az \Rightarrow z = A^{-1}(x-\mu)$

$$\ominus \int_B f_Z(A^{-1}(z-\mu))\,|A^{-1}|\,dz.$$

$\uparrow$ Jacobian = determinant

Hence: $f_X(z) = f_Z(A^{-1}(z-\mu)) \cdot |A^{-1}|$

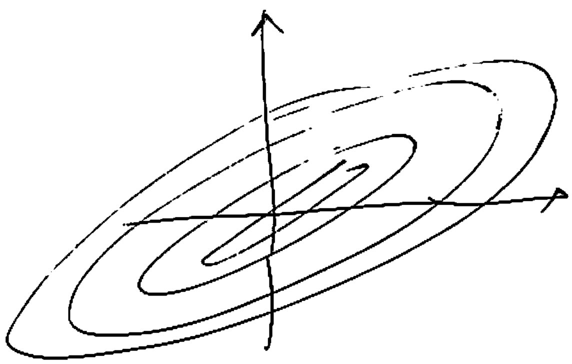
$$= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\|A^{-1}(z-\mu)\|^2} \cdot |A^{-1}|$$

Now, $|A^{-1}| = \frac{1}{|A|} = \frac{1}{|\Sigma|^{1/2}}$ $\left(\text{since } \Sigma = A^2 \Rightarrow |\Sigma| = |A|^2\right)$

$$\|A^{-1}(z-\mu)\|^2 = \langle A^{-1}(z-\mu), A^{-1}(z-\mu)\rangle$$

$$= \left[A^{-1}(z-\mu)\right]^T A^{-1}(z-\mu) = (z-\mu)^T \underbrace{\underbrace{(A^{-1})^T}_{= A^{-1}} A^{-1}}_{= A^{-2} = \Sigma^{-1}} (z-\mu).$$

$\square$



level sets of $f(x)$.