$\boxed{03/14}$

3.7. Conditional distributions & Expectations

X, Y: r.v's (dependent). Conditional distr. of X given $Y=y$? Condit. expectation?

Suppose we know $Y = y$. ~~What does this say about X?~~

Ex. X = height, Y = age. Distribution of weight of 10-y.o. boys? $X \mid Y=10$?

~~Ex: X = height, Y = weight~~

~~$P\{X \in A \mid Y = y\}$ "Conditional distr. of X given Y=y".~~

DISCRETE

<u>Def</u> (<u>discrete</u>) <u>Conditional PMF</u>: of X given $Y=y$ is

$$p_{X|Y}(x|y) := P\{X = x \mid Y = y\} = \frac{p(x,y)}{p_Y(y)}$$

$\leftarrow$ joint pmf; $\leftarrow$ pmf of Y.

<u>Conditional expectation</u>:

$$E[X \mid Y = y] = \sum_x x \cdot P\{X = x \mid Y = y\} = \sum_x x \, p_{X|Y}(x|y).$$

<u>Remark</u> If X, Y are independent, $p_{X|Y}(x|y) = p(x)$; $E[X|Y=y] = E(X)$.

<u>Ex</u> Two dice are thrown; $X :=$ larger, $Y :=$ smaller.
Conditional distr. of X given $Y = 4$?

$$p_Y(4) = P\{Y = 4\} = P\{(4,4), (4,5), (5,4), (4,6), (6,4)\} = \frac{5}{36}.$$

$$p_{X|Y}(x|4) = \begin{cases} \frac{1}{36} \Big/ \frac{5}{36} = \frac{1}{5}, & x = 4 \quad (4,4) \\ \frac{2}{36} \Big/ \frac{5}{36} = \frac{2}{5}, & x = 5 \quad (4,5) \text{ or } (5,4) \\ \frac{2}{36} \Big/ \frac{5}{36} = \frac{2}{5}, & x = 6 \quad (4,6) \text{ or } (6,4) \end{cases}$$

$$E[X \mid Y = 4] = \frac{1}{5} \cdot 4 + \frac{2}{5} \cdot 5 + \frac{2}{5} \cdot 6 = \boxed{5.2}$$

<u>Ex</u>: Compute $E[X]$, compare! $(E[X|Y] > E(X))$

Ex [Ross p.100] Let $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$ be independent.

Compute the distribution (pmf) of X given $X+Y=n$.

Proof

$$P\{X=k \mid X+Y=n\} = \frac{P\{X=k,\ X+Y=n\}}{P\{X+Y=n\}}$$

$$= \frac{P\{X=k,\ Y=n-k\}}{P\{X+Y=n\}} = \frac{P\{X=k\} \cdot P\{Y=n-k\}}{\underbrace{P\{X+Y=n\}}_{\text{Poisson}(\lambda+\mu)}}$$

$$= \frac{e^{-\lambda}\lambda^k}{k!} \cdot \frac{e^{-\mu}\mu^{n-k}}{(n-k)!} \cdot \frac{n!}{e^{-(\lambda+\mu)}(\lambda+\mu)^n}$$

$$= \frac{n!}{k!(n-k)!} \frac{\lambda^k \mu^{n-k}}{(\lambda+\mu)^n}$$

$$= \binom{n}{k} \cdot \Big(\underbrace{\frac{\lambda}{\lambda+\mu}}_{= p}\Big)^k \Big(\underbrace{\frac{\mu}{\lambda+\mu}}_{= 1-p}\Big)^{n-k}.$$

$$\Rightarrow \boxed{X \sim \text{Binom}\left(n, \frac{\lambda}{\lambda+\mu}\right)}$$

~~Remark (intuition)~~. **Remark**: Think about intuitive explanation.

Ex [Ghahramani p. 347] "The Exchange Paradox"

There are 2 identical closed envelopes,
one containing twice as much money as the other.
You are asked to choose one of the envelopes for yourself.
You pick an envelop at random, open it, observe the content.
You are given an option to exchange it for the other envelop.
Should you exchange or not?

$Y :=$ amount in the envelop you pick,
$X :=$ in the other envelop.

$P\{X = 2y \mid Y = y\} = P\{X = \frac{y}{2} \mid Y = y\} = \frac{1}{2}$ by symmetry.

$\Rightarrow E[X \mid Y = y] = 2y \cdot \frac{1}{2} + \frac{y}{2} \cdot \frac{1}{2} = \boxed{1.25y} > y$!

Hence you should always exchange!

"It is better to switch to the second envelop no matter how much money you find in the first one".

Resolution of the paradox: Under these assumptions, $E[X] = \infty$

Proof: if $E[X] < \infty$ then by observing a large amount y, you should conclude that the other envelop can not have $2y$ with prob. $\frac{1}{2}$ (but rather with smaller prob.)
$\Rightarrow$ should not swap.

CONTINUOUS X, Y.

- Conditional pdf $f_{X|Y}(x|y) = ?$

Can't condition on the event $\mathbb{P}\{Y=y\}$ that has zero prob!

- Asymptotic analysis:

CDF: $\mathbb{P}\{X \le x \mid Y \in (y, y+h)\} = \dfrac{\mathbb{P}\{X \le x,\ Y \in (y,y+h)\}}{\mathbb{P}\{Y \in (y,y+h)\}}$

$$= \frac{\int_{-\infty}^{x} \int_{y}^{y+h} f_{XY}(u,v)\, dv\, du}{\int_{y}^{y+h} f_Y(v)\, dv}$$

$f_{X,Y}(u,v) \approx f_{XY}(u,y)$ if $v \in (y, y+h)$

$f_Y(v) \approx f_Y(y)$

$$\approx \frac{\int_{-\infty}^{x} h\, f_{X,Y}(u,y)\, du}{h\, f_Y(y)}$$

$$= \int_{-\infty}^{x} \boxed{\frac{f_{X,Y}(u,y)}{f_Y(y)}}\, du.$$

Hence this is the (conditional) density!

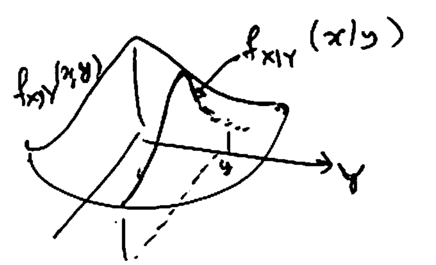
Def (continuous) The conditional PDF of X given $Y=y$ is

$$f_{X|Y}(x|y) := \frac{f_{X,Y}(x,y)}{f_Y(y)},$$

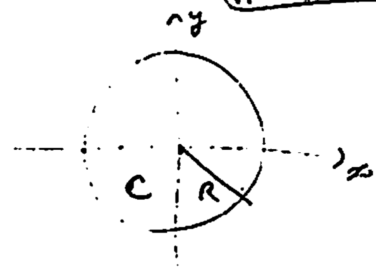
(if $f_Y(y)=0$, define as 0).

Conditional expectation: $\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx$. Function of y!

Remark: Note similarity with discrete case.



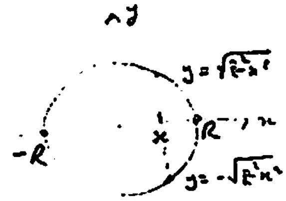
Ex. $(X,Y) \sim \text{Unif}(C)$ where $C = \{(x,y) : x^2+y^2 \le R\}$.



1) Marginal distr. $f_X(x)$, $f_Y(y)$?
2) Are X, Y independent ?
3) Conditional distr. $f_{X|Y}(x|y)$?

Joint pmf.

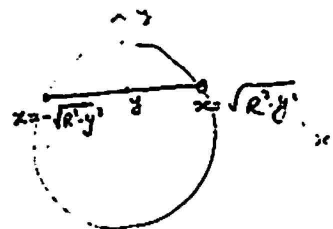
$$f(x,y) = \begin{cases} \frac{1}{\text{Area}(C)} = \frac{1}{\pi R^2}, & x^2+y^2 \le R \\ \text{elsewhere.} \end{cases}$$



i) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}\,dy = \begin{cases} \frac{2\sqrt{R^2-x^2}}{\pi R^2}, & -R \le x \le R \\ 0, & \text{elsewhere} \end{cases}$

$$f_Y(y) = \begin{cases} \frac{2\sqrt{R^2-y^2}}{\pi R^2}, & -R \le y \le R \\ 0, & \text{elsewhere} \end{cases}$$

ii) No, since $f(x,y) \ne f_X(x) \cdot f_Y(y)$



iii) $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{\pi R^2} \cdot \frac{\pi R^2}{2\sqrt{R^2-y^2}}$

$$= \begin{cases} \frac{1}{2\sqrt{R^2-y^2}}, & -\sqrt{R^2-y^2} \le x \le \sqrt{R^2-y^2} \\ 0, & \text{elsewhere.} \end{cases} = \text{Unif}\left[-\sqrt{R^2-y^2}, \sqrt{R^2-y^2}\right]$$

$\therefore X|Y \sim \text{Unif}\left(-\sqrt{R^2-y^2}, \sqrt{R^2-y^2}\right)$

§ 6.5 Conditional distr's : ~~di~~ Continuous case

Def (Cond. distr: continuous). X, Y r.v.'s.

Conditional PDF :

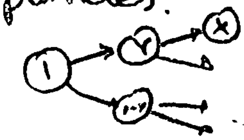
$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

Remark If X, Y are indep. then $f_{X|Y}(x|y) = \frac{f_X(x) f_Y(y)}{f_Y(y)} = f_X(x)$

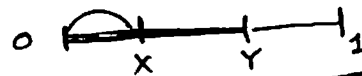
Def (Conditional expectation) (Sect. 7.5.1)

$$E[X|Y=y] = \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y)\, dx$$

Better discuss splitting particles.



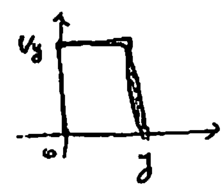
Ex One breaks a stick of length 1 at a uniform random point. Then breaks one of the pieces again. Distr. of the remaining piece ?



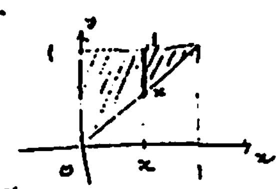
$Y \sim \mathrm{Unif}(0,1)$, $\quad X|Y \sim \mathrm{Unif}(0,Y)$. $\quad$ Distr. of X ?

- We know : $f_Y(y) = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & \text{---} \end{cases}$

$$f_{X|Y}(x|y) = \begin{cases} 1/y, & 0 \le x \le y \\ 0, & \text{---} \end{cases}$$

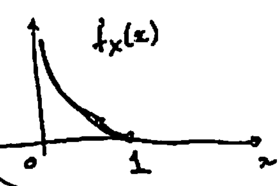


- Joint distr. $\quad f(x,y) = f_{X|Y}(x|y)\, f_Y(y) = \begin{cases} 1/y, & 0 \le x \le y \le 1 \\ 0, & \text{---} \end{cases}$



- Marginal distr $\quad f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_x^1 \frac{1}{y}\, dy = \ln y \Big|_x^1 = -\ln x, \quad 0 \le x \le 1.$

$$f_X(x) = \begin{cases} -\ln x, & 0 \le x \le 1 \\ 0, & \text{---} \end{cases}$$ Not uniform!



Consequence : $E[X] = -\int_0^1 x \ln x\, dx = \left(\frac{1}{4}x^2 - \frac{1}{2}x^2 \ln x\right)\Big|_0^1 = \frac{1}{4}$ ← heuristics $\frac{1}{2} \cdot \frac{1}{2}$

Ex. $E[X|Y=y] = \int x\, f_{X|Y}(x|y)\, dx = \int_0^y \frac{x}{y}\, dx = \frac{1}{y}\frac{y^2}{2} = \boxed{y/2}$ This is clear si . . .

~~Ex Refer to Ex. from 03/21.~~

Ex (Contd)

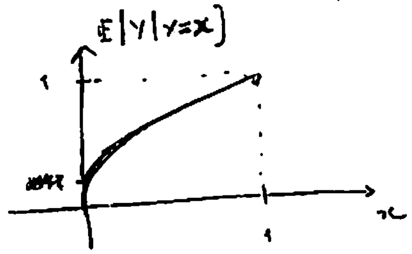
$$\boxed{Y|X \sim ? \qquad E[Y|X=x] = ? \quad \text{(Prediction of } Y \text{ from } X=x\text{).}}$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} . \qquad f(x,y) = \begin{cases} 1/y, & 0 \le x \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

We already computed $\qquad f_X(x) = \begin{cases} -\ln x, & 0 < x \le 1 \\ 0, & \text{elsewhere} \end{cases}$

$$\Rightarrow f_{Y|X}(y|x) = \begin{cases} -\frac{1}{y \ln x}, & x \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$E[Y|X=x] = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy = \int_x^1 \frac{-y}{y \ln x}\, dy = \begin{cases} \frac{1-x}{-\ln x}, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$



For Example: $Y = \underline{0.39}$, $x = 0.12$ $\Rightarrow$ our prediction of Y is $\frac{1-0.12}{-\ln 0.12} \approx \underline{0.42}$.

~~Continuous version of Bayes formula:~~

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y)\, f(y)}{\int_{-\infty}^{\infty} f_{X|Y}(x|y)\, f(y)\, dy} .$$

f_Y = "prior", $f_{Y|X}$ = "posterior" distributions of Y

↑ before / after getting information about the value of X.

In our example, $f_{Y|X}(y|x) = \frac{\frac{1}{y} \cdot 1}{\int_x^1 \frac{1}{y} \cdot 1\, dy} = -\frac{1}{y \ln x}$, $x \le y \le 1$; 0, elsewhere (same answer)

· Another example: Y = # bacteria, X = # white blood cells in blood. Bayes: predict Y given X. (rather, update prediction of Y based on X)

Ex. The lifetime of a bulb is $Exp(\Lambda)$, $\Lambda \sim Unif[a,b]$. Find the (marginal) distr. of the lifetime of the bulb.

$X|\Lambda \sim Exp(\Lambda)$, $\Rightarrow$ $f_{X|\Lambda}(x|\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, \end{cases}$, $f_\Lambda(\lambda) = \frac{1}{b-a}$, $a \le \lambda \le b$.

$\Rightarrow f(x,\lambda) = f_{X|\Lambda}(x|\lambda)\, f_\Lambda(\lambda) = \frac{\lambda}{b-a} e^{-\lambda x}$, $x > 0$. $\Rightarrow f_X(x) = \int_a^b \frac{\lambda}{b-a} e^{-\lambda x}\, d\lambda = \cdots = \frac{e^{-ax}(1+ax) - e^{-bx}(1+bx)}{(b-a)x^2}$

~~- 82 -~~