03/20

Ex 1 $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$ } independent

$P\{X > Y\} = ?$ (wait in one line less than in the other)

~~$P\{X<Y\} = E[P\{X<Y \mid Y\}]$~~

Condition on Y:

$$P\{X > Y\} = \int_0^\infty \underbrace{P\{X > Y \mid Y = y\}}_{} f_Y(y)\, dy$$

$P\{X > y \mid Y = y\} = P\{X > y\}$ (by independence)

$= e^{-\lambda y}$ (since $X \sim Exp(\lambda)$).

$$= \int_0^\infty e^{-\lambda y} \cdot \mu e^{-\mu y}\, dy$$

$$= \mu \cdot \int_0^\infty e^{-(\lambda+\mu) y}\, dy = \boxed{\frac{\mu}{\lambda+\mu}}.$$

Ex 2 A processor consists of n independent cores, each having lifetime $\sim Exp(\lambda_i)$.

(1) What is the expected lifetime of the processor? (it fails if one core fails)

$\min(T_i)$, where $T_i \sim Exp(\lambda_i)$ indep.

$$P\{\min_i T_i > x\} = P\{T_i > x \ \forall i\} = \prod_i P\{T_i > x\} \quad \text{(indep.)}$$

$$= \prod_i e^{-\lambda_i x} = \exp\left[-\left(\sum_1^n \lambda_i\right) x\right]$$

$\Rightarrow \boxed{\min_i T_i \sim Exp\left(\sum_1^n \lambda_i\right)}$

$\Rightarrow E[\min_i T_i] = \dfrac{1}{\sum_1^n \lambda_i}$.

Ex (ctd.) (2) What is the prob. that i'th ~~proc~~ core fails first?
(generalization of Ex1 p.96)

~~$\mathbb{P}\{T_i < \min\}$~~

$$\mathbb{P}\{T_i = \min_j T_j\} = \mathbb{P}\{\underbrace{T_i}_{\sim \text{Exp}(\lambda_i)} < \underbrace{\min_{j \neq i} T_j}_{\sim \text{Exp}(\sum_{j\neq i} T_j)}\}$$ by Ex. p. 96

$$= \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$$

Ex (Ross 5.8) You arrive at a post office which has 2 clerks. Both are currently serving other customers but there is no one else waiting in line. Service times of the clerks are independent Exp(λ_1), Exp(λ_2). ~~Find the expected~~ You will enter service ~~with~~ when either clerk becomes free. Find the expected time you spend in the office. (T)

(Condition on which clerk becomes free first.)

R_1, R_2 := remaining time of the clerks until they become free

~~By memoryless property, $R_1 \sim \text{Exp}(\lambda_1)$, $R_2 \sim \text{Exp}(\lambda_2)$.~~

~~$T = m$~~

S = your service time with a clerk $\Rightarrow$

$$T = \min(R_1, R_2) + S$$

$$\mathbb{E}[T] = \mathbb{E}[\min(R_1, R_2)] + \mathbb{E}[S].$$

By memoryless property, $R_1 \sim \text{Exp}(\lambda_1)$, $R_2 \sim \text{Exp}(\lambda_2)$. $\Rightarrow \min(R_1, R_2) \sim \text{Exp}(\lambda_1 + \lambda_2)$ (Ex. 2 p. 96)

$$\Rightarrow \mathbb{E}[\min(R_1, R_2)] = \frac{1}{\lambda_1 + \lambda_2}.$$

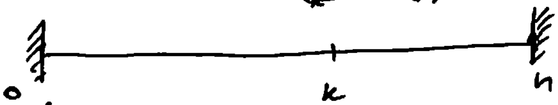
$$\mathbb{E}[S] = \underbrace{\mathbb{E}[S \mid R_1 < R_2]}_{= \frac{1}{\lambda_1}} \underbrace{\mathbb{P}\{R_1 < R_2\}}_{= \frac{\lambda_1}{\lambda_1+\lambda_2}} + \underbrace{\mathbb{E}[S \mid R_2 < R_1]}_{= \frac{1}{\lambda_2}} \underbrace{\mathbb{P}\{R_2 < R_1\}}_{= \frac{\lambda_2}{\lambda_1+\lambda_2}} \quad \text{(LTP)}$$

($\frac{1}{\lambda_1}$: since ~~the condition~~ you are served by clerk 1, so cond. distr. is $\sim \text{Exp}(\lambda_1)$)

($\frac{\lambda_1}{\lambda_1+\lambda_2}$ by Ex. 1 p. 96)

$$\Rightarrow \mathbb{E}[S] = \frac{3}{\lambda_1 + \lambda_2}.$$

Ex (Random walk) Consider a symmetric random walk with absorbing barriers at 0 and n. What is the expected length of walk if we start at k? $=: X_k$



Condition on first step:

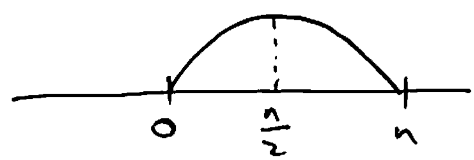
$$\mathbb{E}[X_k] = \underbrace{\mathbb{E}[X_k \mid R]}_{= 1+\mathbb{E}[X_{k+1}]} \cdot \frac{1}{2} + \underbrace{\mathbb{E}(X_k \mid L]}_{= 1+\mathbb{E}(X_{k-1})} \cdot \frac{1}{2}$$

Denoting $\mathbb{E}(X_k) =: E_k$, we find that

$$\begin{cases} E_k = \frac{1}{2} E_{k+1} + \frac{1}{2} E_{k-1} + 1, & k = 1, 2, \dots, n-1 \\ E_0 = E_n = 0 \end{cases}$$

System of linear equations. Solving $\Rightarrow$ (Do this!)

$$E_k = \boxed{k(n-k)}.$$



Remark Maximal at $k = \frac{n}{2}$, in which case expected length of walk $E_{n/2} = \frac{n^2}{4}$. (∗)

Does this make sense?

Related question: how far do we get after t steps?

Displacement $D_t := \sum_{i=1}^{t} Z_i$ where $Z_i = \begin{cases} 1 & \text{prob } 1/2 \\ -1 & \text{prob } 1/2 \end{cases}$ indep.

$\Rightarrow \mathbb{E} D_t = 0$, $Var(D_t) = \sum_1^t \underbrace{Var(Z_i)}_{= \mathbb{E}[Z_i^2] = 1} = t$.

$\Rightarrow$ St. dev. of D_t is $\sqrt{t}$.

Hence: After t steps, we get $\approx \sqrt{t}$ away from $\frac{n}{2}$.

$\Rightarrow$ it takes $t \sim \frac{n^2}{4}$ steps to get to $\approx \sqrt{\frac{n^2}{4}} = \frac{n}{2}$,

i.e. to one of the barriers. Consistent with (∗).

and note that

$$P_k(k) = \prod_{i=1}^{k} p_i, \qquad P_k(0) = \prod_{i=1}^{k} q_i$$

For $0 < j < k$, conditioning on X_k yields the recursion

$$\begin{aligned} P_k(j) &= P\{X_1 + \cdots + X_k = j \mid X_k = 1\}p_k + P\{X_1 + \cdots + X_k = j \mid X_k = 0\}q_k \\ &= P\{X_1 + \cdots + X_{k-1} = j - 1 \mid X_k = 1\}p_k \\ &\quad + P\{X_1 + \cdots + X_{k-1} = j \mid X_k = 0\}q_k \\ &= P\{X_1 + \cdots + X_{k-1} = j - 1\}p_k + P\{X_1 + \cdots + X_{k-1} = j\}q_k \\ &= p_k P_{k-1}(j-1) + q_k P_{k-1}(j) \end{aligned}$$

Starting with $P_1(1) = p_1$, $P_1(0) = q_1$, the preceding equations can be recursively solved to obtain the functions $P_2(j)$, $P_3(j)$, up to $P_n(j)$. ■

a.k.a. "The Secretary Problem".

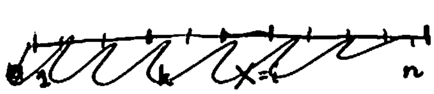
(Ross)

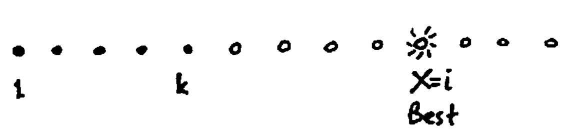
Example 3.23 (The Best Prize Problem) Suppose that we are to be presented with n distinct prizes in sequence. After being presented with a prize we must immediately decide whether to accept it or reject it and consider the next prize. The only information we are given when deciding whether to accept a prize is the relative rank of that prize compared to ones already seen. That is, for instance, when the fifth prize is presented we learn how it compares with the first four prizes already seen. Suppose that once a prize is rejected it is lost, and that our objective is to maximize the probability of obtaining the best prize. Assuming that all $n!$ orderings of the prizes are equally likely, how well can we do?

Solution: Rather surprisingly, we can do quite well. To see this, fix a value $k, 0 \leqslant k < n$, and consider the strategy that rejects the first k prizes and then accepts the first one that is better than all of those first k. Let P_k (best) denote the probability that the best prize is selected when this strategy is employed. To compute this probability, condition on X, the position of the best prize. This gives

$$\begin{aligned} P_k(\text{best}) &= \sum_{i=1}^{n} P_k(\text{best} \mid X = i) P(X = i) \\ &= \frac{1}{n} \sum_{i=1}^{n} P_k(\text{best} \mid X = i) \end{aligned}$$

Now, if the overall best prize is among the first k, then no prize is ever selected under the strategy considered. On the other hand, if the best prize is in position i, where $i > k$, then the best prize will be selected if the best of the first





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k prizes is also the best of the first $i-1$ prizes (for then none of the prizes in positions $k+1, k+2, \ldots, i-1$ would be selected). Hence, we see that

$$\begin{cases} P_k(\text{best}|X=i)=0, & \text{if } i \leqslant k \\ P_k(\text{best}|X=i)=P\{\text{best of first } i-1 \text{ is among the first } k\} \\ \qquad =k/(i-1), & \text{if } i>k. \end{cases}$$

From the preceding, we obtain that

$$\begin{aligned} P_k(\text{best}) &= \frac{k}{n}\sum_{i=k+1}^{n}\frac{1}{i-1} \\ &\approx \frac{k}{n}\int_{k}^{n-1}\frac{1}{x}\,dx \\ &= \frac{k}{n}\log\left(\frac{n-1}{k}\right) \\ &\approx \frac{k}{n}\log\left(\frac{n}{k}\right) \end{aligned}$$

Now, if we consider the function

$$g(x)=\frac{x}{n}\log\left(\frac{n}{x}\right)$$

then

$$g'(x)=\frac{1}{n}\log\left(\frac{n}{x}\right)-\frac{1}{n}$$

and so

$$g'(x)=0 \Rightarrow \log(n/x)=1 \Rightarrow x=n/e$$

Thus, since $P_k(\text{best}) \approx g(k)$, we see that the best strategy of the type considered is to let the first n/e prizes go by and then accept the first one to appear that is better than all of those. In addition, since $g(n/e)=1/e$, the probability that this strategy selects the best prize is approximately $1/e \approx 0.36788$.

Ex (Waiting for a pattern). Toss a coin until we get HT (in a row).
What is the expected # of tosses?

Condition on the first toss.

$$E[X] = E[X|H]\cdot\frac{1}{2} + E[X|T]\cdot\frac{1}{2} \qquad (*)$$

- $E[X|T] = 1 + E[X]$ ("reset")

$E[X|H] = ?$ Condition on the next (second) toss

- $E[X|H] = \underbrace{E[X|HH]}_{= 1+E[X|H] \text{ ("reset")}}\cdot\frac{1}{2} + \underbrace{E[X|HT]}_{= 1 \text{ (success)}}\cdot\frac{1}{2} \xrightarrow{\text{solving}} E[X|H] = 3.$

[illegible]

Substitute into (*) ⇒

$$E[X] = 3\cdot\frac{1}{2} + (1 + E[X])\cdot\frac{1}{2}$$

$\xrightarrow{\text{solving}}$ $E[X] = \boxed{4}$.