Ex 1. \( X \sim \text{Exp} (\lambda), \ Y \sim \text{Exp} (\mu) \) \( \text{independent} \)

\[ P \{ X \leq Y \} = ? \]

(Wait in one line less than in the other)

\[ \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = \frac{1}{\lambda} + \frac{1}{\mu} \]

Condition on \( Y \):

\[
P \{ X \leq Y \} = \int_0^\infty P \{ X \leq y \} f_Y (y) \, dy
\]

\[ P \{ X \leq y \} = e^{-\lambda y} \] \( \text{(since } X \sim \text{Exp}(\lambda)) \)

\[ = e^{-\lambda y} \cdot \mu e^{-\mu y} \, dy \]

\[ = \mu \int_0^\infty e^{-(\lambda + \mu) y} \, dy = \frac{\mu}{\lambda + \mu} \]

Ex 2. A processor consists of \( n \) cores, each having lifetime \( \sim \text{Exp}(\lambda_i) \).

(1) What is the expected lifetime of the processor? (it fails if one core fails)

\[
\min (T_i), \text{where } T_i \sim \text{Exp}(\lambda_i) \text{ indep.}
\]

\[
P \{ \min (T_i) > x \} = \prod_i P \{ T_i > x \} = \prod_i e^{-\lambda_i x} = \exp \left\{ -\left( \sum \lambda \right) x \right\}
\]

\[
E[\min T_i] = \frac{1}{\sum \lambda_i}
\]

\[
E[\text{Exp}(\sum \lambda_i)] = \frac{1}{\sum \lambda_i}
\]
Ex (Ross 5.8) You arrive at a post office which has 2 clerks. Each clerk is currently serving other customers but there is no one else waiting in line. Service times of the clerks are independent $\text{Exp}(\lambda_1)$, $\text{Exp}(\lambda_2)$.

Find the expected time you will enter service when either clerk becomes free. Find the expected time you spend in the office.

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Condition on which clerk becomes free first

$T_i$ is the remaining time of the $i$th clerk until they become free.

By memoryless property, $R_1 \sim \text{Exp}(\lambda_1)$, $R_2 \sim \text{Exp}(\lambda_2)$.

$S = \text{your service time with a clerk} \Rightarrow$

$T = \min(R_1, R_2) + S$

$E[T] = E[\min(R_1, R_2)] + E[S]$.

By memoryless property, $R_1 \sim \text{Exp}(\lambda_1)$, $R_2 \sim \text{Exp}(\lambda_2)$, $\Rightarrow \min(R_1, R_2) \sim \text{Exp}(\lambda_1 + \lambda_2)$ (Exp. 2 p. 96)

$E[S] = E[S | R_1 < R_2] P(R_1 < R_2) + E[S | R_2 < R_1] P(R_2 < R_1)$ (CLT)

$\frac{1}{\lambda_1}$, $\frac{1}{\lambda_2}$ (since the condition you are served by clerk $i$, so $i$ stays, distr. is $\sim \text{Exp}(\lambda_i)$)

$\frac{\lambda_1}{\lambda_1 + \lambda_2}$ (by Exp. 1 p. 96)

$\frac{\lambda_2}{\lambda_1 + \lambda_2}$

$E[S] = \frac{3}{\lambda_1 + \lambda_2}$
(Random walk) Consider a symmetric random walk with absorbing barriers at 0 and n. What is the expected length of walk if we start at k?

Condition on first step:

\[ E[X_k] = \frac{E[X_k | R]}{2} + \frac{E[X_k | L]}{2} \cdot \frac{1}{1 + E[X_{k+1}]} \]

Denoting \( E[X_k] = E_k \), we find that

\[
\begin{align*}
E_k &= \frac{1}{2} E_{k+1} + \frac{1}{2} E_{k-1} + 1, & k &= 1, 2, \ldots, n-1 \\
E_0 &= E_n = 0
\end{align*}
\]

System of linear equations. Solving \( \Rightarrow \)

\[ E_k = k(n-k) \]

Remark: Maximal at \( k = \frac{n}{2} \), in which case the expected length of walk \( E_{\frac{n}{2}} = \frac{n^2}{2} \).

Does it make sense?

Related question: How far do we get after \( t \) steps?

\[ \text{Displacement } D_t := \sum_{i=1}^{t} Z_i \text{ where } Z_i = \begin{cases} 1 \text{ with probability } \frac{1}{2} \\ -1 \text{ with probability } \frac{1}{2} \end{cases} \text{ independent.} \]

\[ \Rightarrow \text{ Cov}(D_t) = t \text{ Var}(Z_i) = t, \quad E[1] = 1 \]

\[ \Rightarrow \text{ St. dev. of } D_t \text{ is } \sqrt{T} \]

Hence: After \( t \) steps, we get \( \pm \sqrt{T} \) away from \( \frac{n}{2} \).

\[ \Rightarrow \text{ it takes } t \left( \frac{n^2}{4} \right) \text{ steps to get to } \pm \sqrt{T} = \frac{n}{2}, \]

i.e. to one of the barriers. Consistent with (4).
and note that
\[ P_k(k) = \prod_{i=1}^{k} p_i, \quad P_k(0) = \prod_{i=1}^{k} q_i \]

For \( 0 < j < k \), conditioning on \( X_k \) yields the recursion
\[
P_k(j) = \sum_{\ell=1}^{j-1} P(X_1 + \cdots + X_{k-1} = \ell | X_k = 1) p_k + P(X_1 + \cdots + X_{k-1} = j | X_k = 0) q_k
\]
\[ = p_k P_{k-1}(j-1) + q_k P_{k-1}(j) \]

Starting with \( P_1(1) = p_1, \ P_1(0) = q_1 \), the preceding equations can be recursively solved to obtain the functions \( P_2(j), P_3(j), \ldots, P_n(j) \).

**Example 3.23** (The Best Prize Problem) Suppose that we are to be presented with \( n \) distinct prizes in sequence. After being presented with a prize we must immediately decide whether to accept it or reject it and consider the next prize. The only information we are given when deciding whether to accept a prize is the relative rank of that prize compared to ones already seen. That is, for instance, when the fifth prize is presented we learn how it compares with the first four prizes already seen. Suppose that once a prize is rejected it is lost, and that our objective is to maximize the probability of obtaining the best prize. Assuming that all \( n! \) orderings of the prizes are equally likely, how well can we do?

**Solution:** Rather surprisingly, we can do quite well. To see this, fix a value \( k, 0 \leq k < n \), and consider the strategy that rejects the first \( k \) prizes and then accepts the first one that is better than all of those first \( k \). Let \( P_k(\text{best}) \) denote the probability that the best prize is selected when this strategy is employed. To compute this probability, condition on \( X \), the position of the best prize. This gives
\[
P_k(\text{best}) = \sum_{i=1}^{n} P_k(\text{best}|X = i) P(X = i)
\]
\[ = \frac{1}{n} \sum_{i=1}^{n} P_k(\text{best}|X = i) \]

Now, if the overall best prize is among the first \( k \), then no prize is ever selected under the strategy considered. On the other hand, if the best prize is in position \( i \), where \( i > k \), then the best prize will be selected if the best of the first
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$k$ prizes is also the best of the first $i - 1$ prizes (for then none of the prizes in positions $k + 1, k + 2, \ldots, i - 1$ would be selected). Hence, we see that

\[
\begin{align*}
P_k(\text{best} | X = i) &= 0, \quad \text{if } i \leq k \\
P_k(\text{best} | X = i) &= P(\text{best of first } i - 1 \text{ is among the first } k) \\
&= k / (i - 1), \quad \text{if } i > k.
\end{align*}
\]

From the preceding, we obtain that

\[
P_k(\text{best}) = \frac{k}{n} \sum_{i=k+1}^{n} \frac{1}{i - 1}
\]

\[
\approx \frac{k}{n} \int_{k}^{n-1} \frac{1}{x} \, dx
\]

\[
= \frac{k}{n} \log \left( \frac{n - 1}{k} \right)
\]

\[
\approx \frac{k}{n} \log \left( \frac{n}{k} \right)
\]

Now, if we consider the function

\[
g(x) = \frac{x}{n} \log \left( \frac{n}{x} \right)
\]

then

\[
g'(x) = \frac{1}{n} \log \left( \frac{n}{x} \right) - \frac{1}{n}
\]

and so

\[
g'(x) = 0 \Rightarrow \log(n/x) = 1 \Rightarrow x = n/e
\]

Thus, since $P_k(\text{best}) \approx g(k)$, we see that the best strategy of the type considered is to let the first $n/e$ prizes go by and then accept the first one to appear that is better than all of those. In addition, since $g(n/e) = 1/e$, the probability that this strategy selects the best prize is approximately $1/e \approx 0.36788$. 

-100-
Ex: (Waiting for HT). Toss a coin until we get HT (in a row).

What is the expected # of tosses?

Condition on the first toss:

\[ E[X] = \frac{1}{2} E[X|H] \cdot \frac{1}{2} + E[X|T] \cdot \frac{1}{2} \] (*)

\[ E[X|T] = 1 + E[X] \quad (\text{"reset"}) \]

\[ E[X|H] = ? \quad \text{Condition on the next (second) toss} \]

\[ E[X|H] = E[X|HH] \cdot \frac{1}{2} + E[X|HT] \cdot \frac{1}{2} \]

\[ 1 + E[X|H] \quad (\text{"reset"}) \]

Solving \[ E[X|H] = 3. \]

Substitute into (*):

\[ E[X] = \left( \frac{1}{2} E[X|H] \cdot \frac{1}{2} + (1 + E[X]) \cdot \frac{1}{2} \right) + 1 \]

Solving \[ E[X] = 4. \]