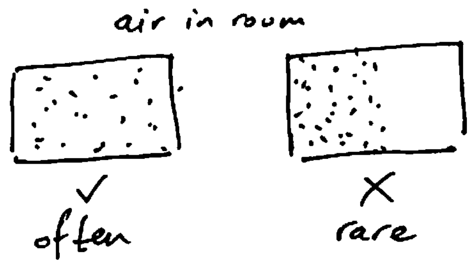
$\boxed{03/30}$

Applications of LLN, CLT.

LLN:

① justifies diffusion:



② <u>Monte-Carlo Integration</u>

Numeric integration $\int_0^1 f(x)\,dx = ?$

~~Let $X_1, X_2, \dots, X_n \sim \text{Unif}[0,1]$ i.i.d. r.v's. Then~~

Interpret $\int_0^1 f(x)\,dx = \mathbb{E}[f(X)]$ where $X \sim \text{Unif}[0,1]$

Let $X_1, \dots, X_n \sim \text{Unif}[0,1]$ i.i.d. r.v's. Strong LLN $\Rightarrow$

$$\boxed{\frac{f(X_1)+\dots+f(X_n)}{n} \to \int_0^1 f(x)\,dx \quad \text{as } n\to\infty \text{ almost surely.}}$$

computable!

(Randomized algorithm).

- Very general!
- Similarly $\forall$ domain $D \subset \mathbb{R}^n$

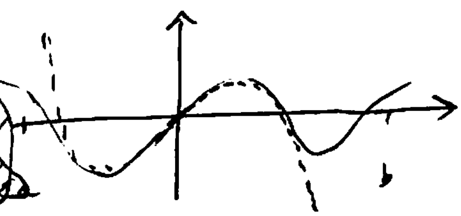
$$\int_D f(x)\,dx.$$

using $X_i \sim \text{Unif}(D)$.

③ Probabilistic proof of Weierstrass Approximation Thm.

"$\forall$ continuous function on a closed interval can be uniformly approximated by a polynomial."

Remark: Can be non-differentiable $\Rightarrow$ Taylor won't work.



Thm (W.A.T.) Let $f : [0,1] \to \mathbb{R}$ be a continuous function.
Then $\exists$ a sequence of polynomials $p_n(x)$ s.t.

$$\underbrace{\| p_n - f \|_\infty}_{\text{"sup-norm"}} := \sup_{x \in [0,1]} | p_n(x) - f(x) | \longrightarrow 0 \quad \text{as } n \to \infty.$$

Informal Proof : Fix $x \in [0,1]$ and consider

$X_1, X_2, \ldots, X_n \sim \mathrm{Ber}(x)$;

$S_n := X_1 + \cdots + X_n \sim \mathrm{Binom}(n, x)$.

LLN: $\frac{S_n}{n} \approx x$

$\Rightarrow f\left(\frac{S_n}{n}\right) \approx f(x)$

$\Rightarrow \mathbb{E}\left[f\left(\frac{S_n}{n}\right)\right] \approx f(x)$

$\parallel$ $S_n \sim \mathrm{Binom}(n, x)$

$$\boxed{\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} =: p_n(x)}, \quad \text{polynomial}.$$

Called "Bernstein's polynomials".

- Simple, constructive approximation!

Formal proof of W.A.T. : Fix $\delta > 0$;

- WLLN $\Rightarrow \mathbb{P}\left\{\left|\frac{S_n}{n} - x\right| \geq \delta\right\} \to 0$ as $n \to \infty$.

  We claim that this convergence is <u>uniform in x</u>.

  This follows from the proof of WLLN:

$$\mathbb{P}\left\{\left|\frac{S_n}{n} - x\right| \geq \delta\right\} \leq \frac{\mathrm{Var}(S_n/n)}{\delta^2} \quad \text{(Chebyshev)}$$

$$= \frac{n x(1-x)/n^2}{\delta^2} \leq \frac{1}{n\delta^2} \quad (\text{since } x \in [0,1]) \qquad (☺)$$

  Hence convergence is uniform in x.

- $f : [0,1] \to \mathbb{R}$ is continuous $\Rightarrow$ <u>uniformly continuous</u>: (Heine-Cantor Thm)

  $\forall \varepsilon > 0 \ \exists \delta > 0$ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. $(*)$

  Moreover, f is ~~unifor~~ <u>bounded</u>:

  $\exists M : \ |f(x)| \leq M \quad \forall x \in [0,1]$. $(**)$

- We want to bound

$$|p_n(x) - f(x)| = \left|\mathbb{E}\left[f\left(\frac{S_n}{n}\right)\right] - f(x)\right| = \Big|\mathbb{E}\,\underbrace{\left[f\left(\frac{S_n}{n}\right) - f(x)\right]}_{\overset{!!}{R_n} \ (\text{r.v.})}\Big|$$

  $\uparrow$ Bernstein polynomial

$$\leq \mathbb{E}|R_n| \quad \text{(Jensen)}$$

$$= \mathbb{E}\,\underbrace{|R_n|}_{\overset{\wedge}{\varepsilon}\text{ by } (*) \text{ \& def of } R_n} \cdot \underbrace{\mathbb{1}_{\{|S_n/n - x| < \delta\}}}_{\leq 1} + \mathbb{E}\,\underbrace{|R_n|}_{\overset{\wedge}{2M} \text{ by } (**)} \cdot \underbrace{\mathbb{1}_{\{|S_n/n - x| \geq \delta\}}}_{\leq 1}$$

$$< \varepsilon + 2M \cdot \mathbb{P}\{|S_n/n - x| \geq \delta\}$$

$$< \varepsilon + \frac{2M}{n\delta^2} \quad (\text{by } ☺).$$

Take $\sup_{x \in [0,1]}$; Let $n \to \infty \Rightarrow$ ~~lim~~ $\lim_{n\to\infty} \|p_n - f\|_\infty \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\lim_n \|p_n - f\|_\infty = 0$.

—///—

CLT:

1. Explains prevalence of normal distribution in nature, technology, etc

2. Explains why Brownian motion is ~ normal.
(random walk)

DEVIATION INEQUALITIES.

Drawback of CLT: does not guarantee good speed of convergence.
(error of approximation typically larger than the tail)

Concentration/deviation inequalities.

$$\boxed{\begin{array}{l} \textbf{Thm (Hoeffding's ineq.)}\ \text{Let } X_1,\dots,X_n \text{ be } \overset{\text{iid}}{\cancel{\text{indep.}}} \text{ r.v.'s with mean } 0 \\ \text{and such that } |X_i| \le 1 \text{ a.s.} \quad \cancel{\text{Then, } \forall t>0}: \\ \text{Let } S_n = X_1 + \dots + X_n. \quad \text{Then } \forall a>0: \\ \qquad \mathbb{P}\{|S_n| > a\} \le 2e^{-a^2/2n} \end{array}}$$

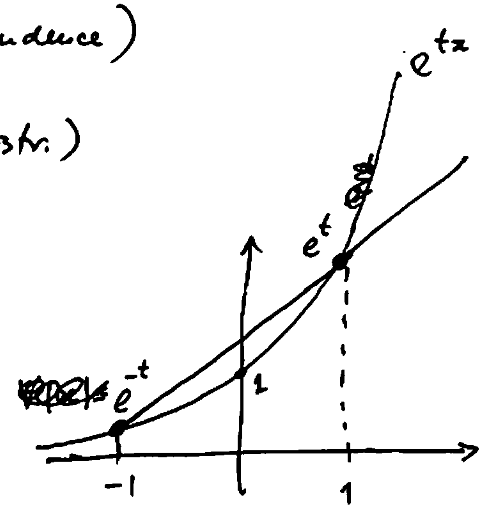
Based on MGF: ~~Choos~~ Let $t>0$ be $\forall$; then

- $\mathbb{P}\{S_n > a\} = \mathbb{P}\{e^{tS_n} > e^{ta}\}$

$$\le e^{-ta}\,\mathbb{E}[e^{tS_n}] \qquad \text{(Markov)}$$

$$= e^{-ta}\prod_{i=1}^{n}\mathbb{E}[e^{tX_i}] \qquad \text{(independence)}$$

$$= e^{-ta}\left(\mathbb{E}[e^{tX_1}]\right)^n \qquad \text{(identical distr.)}$$



- $\mathbb{E}[e^{tX_1}] \le ?$

Function $f(x) := e^{tx}$ is convex $\Rightarrow$

$$e^{tx} \le \left(\frac{e^t + e^{-t}}{2}\right) + \left(\frac{e^t - e^{-t}}{2}\right)x$$

$$\forall\ -1 \le x \le 1$$

Let $x = X_1$; take $\mathbb{E}$ of both sides; use $\mathbb{E}[X_1] = 0$

$$\mathbb{E}[e^{tX_1}] \le \frac{e^t + e^{-t}}{2} \le e^{t^2/2} \quad \forall t > 0 \qquad \text{(Compare Taylor's expansions)}$$

$$\Rightarrow \mathbb{P}\{S_n > a\} \le e^{-ta} \cdot e^{nt^2/2} = \exp\left(-ta + nt^2/2\right).$$

- Optimize in t $\left(t := a/n\right) \Rightarrow$

$$\mathbb{P}\{S_n > a\} \le e^{-a^2/2n}.$$

- Repeat for $-S_n$, take union, qed.

Remark Normalizing like in CLT: $\frac{S_n}{\sqrt{n}}$ has mean 0, var 1

$$\mathbb{P}\left\{ \left|\frac{S_n}{\sqrt{n}}\right| > x \right\} \le \mathbb{P}\left\{ |S_n| \ge \underset{\overset{\shortparallel}{a}}{x\sqrt{n}} \right\} \le 2e^{-x^2/2}$$

just like $N(0,1)$

"Non-asymptotic" version of CLT.

Remark If $|X_i| \le M$ then $\mathbb{P}\{|S_n| > a\} \le 2\exp\left(-\frac{a^2}{2M^2 n}\right)$ (check!)

If $|X_i - \underset{\overset{\shortparallel}{\mathbb{E}X_i \ne 0}}{\mu}| \le M$ still OK. (why?)

~~Ex (refer to Ex p. (07)). Koeffding gives (with $|X_i - \overset{\mathbb{E}X_i}{1}| \le 1$)~~

~~$\mathbb{P}\{S_n \ge 320\} \le \mathbb{P}\{S_n - 300 \ge 20\} \le \exp\left(-\frac{20^2}{2 \cdot 300}\right)$~~

~~Ex. Roll a die n times, record total sum X~~

~~$\mathbb{P}\{X \ge \frac{7n}{2}\}$ $S_n = X_1 + \cdots + X_n$, $\mathbb{E}[X_i] = \frac{7}{2}$, $\mathrm{Var}(X_i) = \frac{35}{12}$~~

~~For large n, $\mathbb{P}\left\{ \left|S_n - \frac{7n}{2}\right| > 5\sqrt{n} \right\} \approx 0.034$ from CLT (Ex)~~

~~≤ 0.2~~

Ex (Markov, Chebyshev, Koeffding, CLT).

Flip a fair coin 1,000 times.
What is the probability of more than 600 heads?

$\left(\mathbb{E}[S_n] = n/2, \ \mathrm{Var}(S_n) = n/4, \ n = 1{,}000 \right)$

- Markov: $p_n := \mathbb{P}\{S_n > 0.6n\} \le \frac{\mathbb{E}[S_n]}{0.6n} = \frac{0.5n}{0.6n} = \frac{5}{6}$
- Chebyshev: $p_n \le \mathbb{P}\{|S_n - \frac{n}{2}| > 0.1n\} \le \frac{\mathrm{Var}(S_n)}{0.01n^2} = \frac{n/4}{0.01n^2} = \frac{25}{n}$ (Better)
- Koeffding: $p_n \le \exp\left(-\frac{0.01n^2}{2n}\right) = \exp(-n/200)$ (Better for large n).
- CLT: $p_n = \mathbb{P}\left\{ \frac{S_n - n/2}{\sqrt{n/4}} > \frac{0.1n}{\sqrt{n/4}} \right\} \overset{CLT}{\simeq} \mathbb{P}\{\underset{N(0,1)}{Z} > 0.2\sqrt{n}\} = 1 - \Phi(0.2\sqrt{n})$ (Better than Koeffding for small n)