04/11

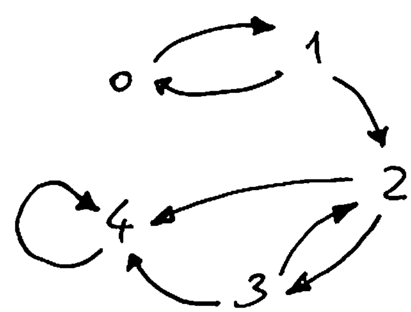
## 3. Classificatication of states

(Ross 4.3, Walsch 7.6)

**Recall**
- $P = (P_{ij})$ transition probability matrix
  $$P_{ij} = \mathbb{P}\{X_{k+1} = j \mid X_k = i\} \qquad (\forall k)$$
- $P^n$ : n-step transition prob matrix
  $$(P^n)_{ij} = \mathbb{P}\{X_{k+n} = j \mid X_k = i\}$$
  (Chapman-Kolmogorov eq's)

**Def** State $j$ is accessible from state $i$ if
$$\exists n : (P^n)_{ij} > 0.$$
States $i, j$ communicate (denoted $i \leftrightarrow j$) if $j$ is accessible from $i$ & vice versa.

**Ex**
$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & \frac{2}{5} & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



"Transition graph"

- 2 is accessible from 0 but not from 4.
- 2, 3 communicate.

4 is "absorbing state"

**Prop**
(i) $i \leftrightarrow i$ ("reflexivity")
(ii) if $i \leftrightarrow j$ then $j \leftrightarrow i$ ("symmetry")
(iii) if $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$. ("transitivity")
In other words, communication is an equivalence relation on the states.

<u>Proof</u> (i) for $n=0$ trivial (we may define $P^0 := I_n$) ← identity

(ii) follows by def ~~symmetry of P, thus $P^n$~~.

(iii) Enough to show: if $j$ accessible from $i$, $k$ accessible from $j$, then $k$ accessible from $i$.

So, assume $(P^n)_{ij} > 0$, $(P^k)_{jk} > 0$

$$\begin{tikzcd} i \arrow[r, bend left, "(P^n)_{ij}"] \arrow[rr, dashed, bend right] & j \arrow[r, bend left, "(P^k)_{jk}"] & k \end{tikzcd}$$

$$(P^{n+k})_{ik} = \sum_s (P^n)_{is}(P^k)_{sk} \qquad \left[\text{since } \cancel{\text{Chapman-Kolm.}}\ P^{n+k} = P^n \cdot P^k\right]$$

$$\geq \underset{\overset{\vee}{0}}{(P^n)_{ij}} \underset{\overset{\vee}{0}}{(P^k)_{jk}} > 0.$$

QED. ⌟

<u>Remark</u> ~~Recap~~ Recall: an equivalence relation on a set $S$ divides $S$ into a collection of disjoint subsets called "<u>equivalence classes</u>".

<u>Example</u>: define $x \sim y$ if $x-y$ is even number, for $x, y \in \mathbb{Z}$. Then the equivalence classes are: {even numbers}, {odd numbers}.

$$\boxed{\begin{minipage}{0.9\textwidth}\end{minipage}}$$

<u>Cor</u> The set of states can be divided into a collection of disjoint classes. Each class contains all states that communicate with each other; none of states from <u>different</u> classes communicate.

<u>Ex</u> (a) from p. 115: $\{0,1\}$, $\{2,3\}$, $\{4\}$ ~~[illegible]~~.

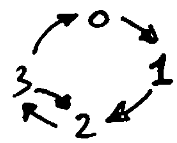
(b) Symmetric random walk with absorbing barriers $0, n$: $\{0, \ldots n\}$

Classes: $\{0\}$, $\{1, \ldots, n-1\}$, $\{n\}$.

**Def** A Markov chain with only one class is called <u>irreducible</u>.

i.e. all states communicate.

**Ex.** $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \end{bmatrix}$




irreducible.

## Recurrent and Transient States.

**Def** Fix a state $i$, and consider the probability

$$p := \mathbb{P}\{\text{starting in } i, \text{ the process will ever return to } i\}.$$

If $p=1$, $i$ is called a <u>recurrent</u> state.

If $p<1$, $i$ is called a transient state.

(More formally: define <u>hitting time</u>

$$T := \inf\{n : X_n = i\}$$

Then $$p = \mathbb{P}\{T < \infty \mid X_0 = i\}.$$ )

**Prop** (# of returns) (i) If state $i$ is recurrent then, with probability 1, ("a.s.") the process will return to $i$ <u>infinitely</u> many times.

(ii) If state $i$ is transient, then, with probability 1, ("a.s") the process will return to $i$ <u>finitely</u> many times (i.e. will eventually leave $i$ and never return).

(Formally, define <u># of returns</u> $N := \#\{n \geq 1 : X_n = i\}$.

(i) $i$ recurrent $\Rightarrow \mathbb{P}\{N = \infty \mid X_0 = i\} = 1$

(ii) $i$ transient $\Rightarrow \mathbb{P}\{N < \infty \mid X_0 = i\} = 1$. )

**Proof** (ii)

$\mathbb{P}\{\text{ever returns}\} = p$ (def. of recurrent)

$\mathbb{P}\{\text{returns at least twice}\} = \mathbb{P}\{\text{returns again} \mid \text{ever returns}\} \cdot \mathbb{P}\{\text{ever returns}\}$

$= \mathbb{P}\{\text{ever returns}\} \cdot \mathbb{P}\{\text{ever returns}\}$ (by Markov prop)

$= p^2$

$\mathbb{P}\{\text{returns} \geq n \text{ times}\} = p^n \quad \forall n.$

$\Updownarrow$

$\mathbb{P}\{N \geq n\} = p^n$

$\cancel{\mathbb{E}[N] = \sum_{n=1}^{\infty} \mathbb{P}\{N \geq n\} = \sum_{n=1}^{\infty} p^n = \frac{p}{1-p} < \infty}$

~~Hence~~

$\Rightarrow \mathbb{P}\{N = \infty\} \leq \mathbb{P}\{N \geq n\} = p^n \to 0$ as $n \to \infty$

$\Rightarrow \mathbb{P}\{N = \infty\} = 0.$ ∎

(i) similar (do it!)

**Remark** Proof above gives expected # of returns for transient states

$$\mathbb{E}[N]\Big|_{X_0 = i} = \sum_{n=1}^{\infty} p^n = \frac{p}{1-p}.$$

~~**Question:** How to tell if $i$ is recurrent or transient, in terms of transition prob's?~~

~~**Prop** $i$ is recurrent $\iff \sum_{n=1}^{\infty} (P^n)_{ii} = \infty$~~

~~(thus, $i$ is transient $\iff \sum_{n=1}^{\infty} (P^n)_{ii} < \infty$)~~

~~Will show: $\mathbb{E}[N] = \sum_{n=1}$~~

Another useful expression, via transition prob's, is

Lem $\boxed{E[N \mid X_0 = i] = \sum_{n=1}^{\infty} (P^n)_{ii}}$

$\ulcorner$ $N = \sum_{n=1}^{\infty} I_n$ where $I_n := \begin{cases} 1, & \text{if } X_n = i \\ 0, & \text{if } X_n \neq i \end{cases}$.

~~$E[N] = \sum_{n=1}^{\infty} E[$~~

$$E[N \mid X_0 = i] = \sum_{n=1}^{\infty} E[I_n \mid X_0 = i]$$

$$= \sum_{n=1}^{\infty} P\{X_n = i \mid X_0 = i\} = \sum_{n=1}^{\infty} (P^n)_{ii}. \quad \lrcorner$$

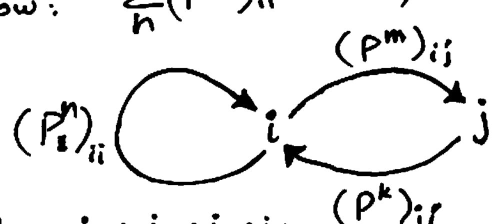
(Recurrence in terms of ~~transition prob~~)

$\Rightarrow$ **Thm** State $i$ is recurrent if $\sum_{n=1}^{\infty} (P^n)_{ii} = \infty$ and is transient if $\sum_{n=1}^{\infty} (P^n)_{ii} < \infty$. $\square$.

**Thm** Recurrence is a class property.
In other words, if a state $i$ is recurrent then all states that communicate with $i$ are recurrent, too.
Same about transient.

$\ulcorner$ Suppose $i$ is recurrent and $i \leftrightarrow j$. WTS: $j$ is recurrent.

We know: $\sum_n (P^n)_{ii} = \infty$ ; WTS: $\sum_n (P^n)_{jj} = \infty$




By assumption ($i \leftrightarrow j$), $\exists\, m, n$: $(P^m)_{ij} > 0$, $(P^k)_{ji} > 0$.

Then, walking $j \to i \to i \to j$:

$$(P^{k+n+m})_{jj} \geq (P^k)_{ji} (P^n)_{ii} (P^m)_{ij}. \qquad \text{Sum up:}$$

$$\sum_{n=1}^{\infty} (P^{k+n+m})_{jj} \geq \underbrace{(P^k)_{ji}}_{>0} \underbrace{(P^m)_{ij}}_{>0} \underbrace{\sum_{n=1}^{\infty} (P^n)_{ii}}_{=\infty} = \infty.$$

$$\wedge\!\!\!\wedge\ \ \sum_{n=1}^{\infty} (P^n)_{jj}.$$

$\square$. $\lrcorner$

**Cor** (i) In a finite Markov chain, $\exists$ a recurrent class.
(ii) In particular, in a finite irreducible Markov chain, all states are recurrent.

(i) Otherwise, the process will eventually leave all states (by Prop. p. 117), which is impossible.
(ii) directly from (i).

**Prop** Once the process enters a recurrent class $R$, it stays in $R$ forever with prob. 1.
("$\forall$ recurrent class is absorbing")

Suppose not: $\exists$ states $i \in R$, $j \notin R$: <u>$j$ is accessible from $i$.</u>


$i$ is recurrent,
thus the process must return from $j$ to $i$ with prob 1,
hence <u>$i$ is accessible from $j$</u>. Hence $i \leftrightarrow j$,
so $j$ must be in the same class $R$ as $i$. $\lightning$

**Remark** In other words, if $i \in$ Recurrent class, while $j$ is NOT,
we must have $P_{ij} = 0$. ~~(They can't communicate!)~~

**Ex. p. 110**: Useful to study <u>classes</u> not states:

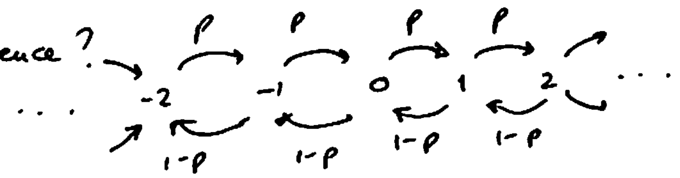
(0, 1) $\longrightarrow$ (2, 3) $\longrightarrow$ (4)

TRANSIENT (Prop. above) — TRANSIENT (Prop. above) — RECURRENT. (Cor. above)

**Ex p. 117**:  All states recurrent (it is irreducible: all states communicate; thus by Cor. above the only class is recurrent).

# Application : random walks.

- Recurrence ?




$p \in (0,1)$

- Irreducible Markov chain.

  Hence, all states are either recurrent or transient.

  Question: which ?

$$\sum_{n=1}^{\infty} (P^n)_{00} \begin{cases} = \infty \,? & (\text{recur}) \\ < \infty \,? & (\text{trans.}) \end{cases}$$

- Odd: $(P^{2n-1})_{00} = 0 \quad \forall n$, since it takes an even # of steps to go back.

- Even: $(P^{2n})_{00} = \binom{2n}{n} p^n (1-p)^n$

  deciding which steps are to the right / left

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$$

- Use Stirling's formula $\boxed{n! \sim n^n e^{-n} \sqrt{2\pi n}}$

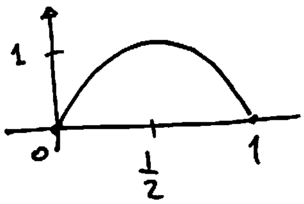
  where $a_n \sim b_n$ stands for $\frac{a_n}{b_n} \to 1$ as $n \to \infty$.

$$\Rightarrow \quad \binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}.$$

- Thus $(P^{2n})_{00} \sim \dfrac{[4p(1-p)]^n}{\sqrt{\pi n}}$.

- By ~~ratio~~ limit ratio test, the series $\sum_{n=1}^{\infty} (P^n)_{00}$ converges iff $\sum_{n=1}^{\infty} \dfrac{[4p(1-p)]^n}{\sqrt{\pi n}}$ converges

$4p(1-p)$




- If $p = \frac{1}{2}$, $\sum_{n\geq 1}^{\infty} \frac{1}{\sqrt{\pi n}} = \infty$
- If $p \neq \frac{1}{2}$, $4p(1-p) \in (0,1) \Rightarrow$

$$\sum_{n\geq 1}^{\infty} \frac{[4p(1-p)]^n}{\sqrt{\pi n}} < \infty.$$

Thus, we proved:

$\boxed{\text{\underline{Thm} A simple random walk on } \mathbb{Z} \text{ is recurrent if } p = \tfrac{1}{2} \text{ and is transient if } p \neq \tfrac{1}{2}.}$

<u>Remark</u> (Higher dimensions; $p = \frac{1}{2}$)

- $d = 2$: recurrent.
- $d \geq 3$: transient.