Recall \( P = (p_{ij}) \) transition probability matrix
\[ p_{ij} = P \{ X_{k+1} = j \mid X_k = i \} \quad (\forall k) \]
- \( P^n \): \( n \)-step transition prob matrix
\[ (P^n)_{ij} = P \{ X_{k+n} = j \mid X_k = i \} \]

(Chapman-Kolmogorov)
Eq's

Definition: State \( j \) is accessible from state \( i \) if
\[ \exists n : (P^n)_{ij} > 0. \]
States \( i, j \) communicate (denoted \( i \leftrightarrow j \)) if
\( j \) is accessible from \( i \) & vice versa.

Example
\[
P = \begin{bmatrix}
0 & \frac{1}{3} & 0 & \frac{1}{4} & 0 \\
\frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\
0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\
0 & 0 & 0 & \frac{2}{3} & 0
\end{bmatrix}
\]

- 2 is accessible from 0 but not from 4.
- 2, 3 communicate.
- 4 is "absorbing state".

"Transition graph".

Propositions
(i) \( i \leftrightarrow i \) ("reflexivity")
(ii) if \( i \leftrightarrow j \) then \( j \leftrightarrow i \) ("symmetry")
(iii) if \( i \leftrightarrow j \) and \( j \leftrightarrow k \) then \( i \leftrightarrow k \). ("transitivity")

In other words, communication is an equivalence relation on the states.
Proof

(i) for n=0 trivial (we may define $P^0 = I_n$) def

(ii) follows by symmetry of $P^n$.

(iii) Enough to show: if $j$ accessible from $i$, $k$ accessible from $j$, then $k$ accessible from $i$.

So, assume $(P^n)^{ij} > 0$, $(P^k)^{jk} > 0$

\[
(P^n)^{ij} \xrightarrow{i} j \xrightarrow{j} k
\]

\[
(P^{n+k})^{ik} = \sum_{s} (P^n)^{is}(P^k)^{sk}
\]

\[
\geq (P^n)^{ij}(P^k)^{jk} > 0.
\]

\[\text{since}\]

\[
(P^{n+k}) = P^n P^k
\]

Q.E.D.

Remark: Recall: an equivalence relation on a set $S$ divides $S$ into a collection of disjoint subsets called "equivalence classes".

Example: define $x \sim y$ if $x - y$ is even number, for $x, y \in \mathbb{Z}$.

Then the equivalence classes are: \{even numbers\}, \{odd numbers\}.

Cor. The set of states can be divided into a collection of disjoint classes. Each class contains all states that communicate with each other; none of states from different classes communicate.

Ex(a) from p. 115: {10, 11}, {2, 3, 5}, {4}

(b) Symmetric random walk with absorbing barriers on $\{0, \ldots, n\}$

Classes: {0, 3, 1, ..., n-1}, {1}. 
A Markov chain with only one class is called **irreducible**, i.e. all states communicate.

\[
P = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0
\end{bmatrix}
\]

**irreducible**

**Ex.**

**Recurrent and Transient States.**

**Def.** Fix a state \( i \), and consider the probability

\[
p = \mathbb{P}\{ \text{starting in } i, \text{ the process will ever return to } i \}.
\]

- If \( p=1 \), \( i \) is called a **recurrent state**.
- If \( p<1 \), \( i \) is called a **transient state**.

(More formally: define **hitting time**)

\[
T := \inf \{ n : X_n = i \}
\]

Then

\[
p = \mathbb{P}\{ T < \infty \mid X_0 = i \}.
\]

**Prop.** (\# of returns)

(i) If state \( i \) is recurrent, then, with probability 1, \( (a.s.) \)

the process will return to \( i \) infinitely many times.

(ii) If state \( i \) is transient, then, with probability 1, \( (a.s.) \)

the process will return to \( i \) finitely many times

(i.e. will eventually leave \( i \) and never return).

(Formally, define \# of returns) \( N := \# \{ n \geq 1 : X_n = i \} \).

(i) \( i \) recurrent \( \Rightarrow \) \( \mathbb{P}\{ N = \infty \mid X_0 = i \} = 1 \)

(ii) \( i \) transient \( \Rightarrow \) \( \mathbb{P}\{ N < \infty \mid X_0 = i \} = 1 \).
Proof

\[ P \{ \text{ever returns} \} = p \quad \text{(def. of recurrent)} \]

\[ P \{ \text{returns at least twice} \} = P \{ \text{returns again after ever returns} \} \]

\[ = P \{ \text{ever returns} \} \cdot P \{ \text{ever returns} \} \quad \text{(by Markov prop)} \]

\[ = p^2 \]

\[ P \{ \text{returns} \geq n \text{ times} \} = p^n \quad \forall n. \]

\[ P \{ N \geq n^2 \} = p^n \]

\[ \Rightarrow P \{ N = \infty \} \leq P \{ N \geq n \} = p^n \to 0 \quad \text{as } n \to \infty \]

\[ \Rightarrow P \{ N = \infty \} = 0. \]

(i) similar (do it!)

Remark: Proof above gives expected \( \# \) of returns for transient states

\[ E[N] = \sum_{n=1}^{\infty} n p^n = \frac{p}{(1-p)^2} \]

Question: How to tell if \( i \) is recurrent or transient in terms of transition prob's:

Prop: \( i \) is recurrent \( \iff \sum_{n=1}^{\infty} (p^n)_{ii} = \infty \)

(thus, \( i \) is transient \( \iff \sum_{n=1}^{\infty} (p^n)_{ii} < \infty \))

Will show: \( E[N] = \sum_{n=1}^{\infty} n (p^n)_{ii} \)
Another useful expression, via transition prob's, is

\[ E[N] = \sum_{n=1}^{\infty} P_{ii}(P^n)_{ii} \]

\[
N = \sum_{n=1}^{\infty} I_n \text{ where } I_n = \begin{cases} 1, & \text{if } X_n = i \\ 0, & \text{if } X_n \neq i \end{cases}
\]

\[
E[N] = \sum_{n=1}^{\infty} \mathbb{E}[I_n]
\]

\[
E[N | X_0 = i] = \sum_{n=1}^{\infty} \mathbb{E}[I_n | X_0 = i]
\]

\[
= \sum_{n=1}^{\infty} P \{ X_n = i | X_0 = i \} = \sum_{n=1}^{\infty} (P^n)_{ii}.
\]

(Recurrence in terms of transition prob.)

\[
\Rightarrow \text{ The state } i \text{ is recurrent if } \sum_{n=1}^{\infty} (P^n)_{ii} = \infty
\]

and is transient if \( \sum_{n=1}^{\infty} (P^n)_{ii} < \infty \).

Recurrence is a class property.

In other words, if a state \( i \) is recurrent then all states that communicate with \( i \) are recurrent, too.

Same about transient.

Suppose \( i \) is recurrent and \( i \rightarrow j \). WTS: \( j \) is recurrent.

We know: \( \sum_{n=1}^{\infty} (P^n)_{ii} = \infty \); WTS: \( \sum_{n=1}^{\infty} (P^n)_{jj} < \infty \)

(\( P^n \))

Then, walking \( j \rightarrow i \rightarrow i \rightarrow j \):

\[
(P^n)_{ij} \geq (P^k)_{ji} (P^n)_{ii} (P^m)_{ij} \quad \text{Sum up:}
\]

\[
\sum_{n=1}^{\infty} (P^{k+m+n})_{jj} \geq (P^k)_{ji} (P^n)_{ii} (P^m)_{ij} \quad \sum_{n=1}^{\infty} (P^n)_{ii} = \infty.
\]

\[
\sum_{n=1}^{\infty} (P^n)_{ij}.
\]
In a finite Markov chain, there exists a recurrent class.

(i) In particular, in a finite irreducible Markov chain, all states are recurrent.

(ii) Otherwise, the process will eventually leave all states (by Prop. p. 117), which is impossible.

(i) directly from (i).

Prop. Once the process enters a recurrent class $R$, it stays in $R$ forever with prob. 1.

(Prop. recurrent class is absorbing)

Suppose not: \[ \exists \text{ states } i \in R, j \notin R: \] \[ j \text{ is accessible from } i. \]

\[ i \text{ is recurrent, } \] \[ \text{Thus the process must return to } i \text{ with prob. 1,} \]

\[ \text{hence } j \text{ is accessible from } i. \]

\[ \text{Hence } i \leftrightarrow j, \]

\[ \text{so } j \text{ must be in the same class } R \text{ as } i. \]

Remark: In other words, if $i \in \text{Recurrence class}$ while $j$ is not, we must have $P_{ij} = 0$. (They can't communicate!)

Ex. p. 110: Useful to study classes not states:

- transient (Prop. above)
- transient (Prop. above)
- recurrent (Cor. above)

Ex. p. 117: All states recurrent (it is irreducible; all states communicate; thus by Cor. above the only class is recurrent).
Application: random walks.

- Recurrence?
  \[ \begin{array}{cccc}
  P & P & P & P \\
  \cdots & -2 & -1 & 0 \\
  & 1-p & 1-p & 1-p \\
  & 1-p & 1-p & 1-p \\
  \end{array} \]
  \[ p \in (0,1) \]

- Irreducible Markov chain.
  Hence, all states are either recurrent or transient.
  Question: which?
  \[ \sum_{n=1}^{\infty} (P^n)_{oo} = \infty ? \quad (\text{recur}) \]
  \[ < \infty ? \quad (\text{trans}) \]

  Odd:
  \[ (P^{2n-1})_{oo} = 0 \quad \forall n, \text{ since it takes an even # of steps to go back.} \]

  Even:
  \[ (P^{2n})_{oo} = \binom{2n}{n} p^n (1-p)^n \]
  \[ \text{deciding which steps are to the right/left} \]

  \[ \binom{2n}{n} = \frac{(2n)!}{(n!)^2} \approx \frac{2^{2n}}{\sqrt{\pi n}} \]

  Use Stirling's formula, where \( a_n \sim b_n \) stands for \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \).

  \[ n! \sim n^n e^{-n} \sqrt{2\pi n} \]

  \[ \Rightarrow \quad \binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}} \]

  Thus:
  \[ (P^{2n})_{oo} \sim \left[ \frac{4p(1-p)}{\sqrt{\pi n}} \right]^n \]

  By limit ratio test, the series \( \sum_{n=1}^{\infty} (P^n)_{oo} \) converges if
  \[ \sum_{n=1}^{\infty} \frac{[4p(1-p)]^n}{\sqrt{\pi n}} \text{ converges.} \]
Thus, we proved:

**Thm.** A simple random walk on \( \mathbb{Z} \) is recurrent if \( p = \frac{1}{2} \) and is transient if \( p > \frac{1}{2} \).

**Remark (Higher dimensions; \( p = \frac{1}{2} \))**

- \( d = 2 \): recurrent.
- \( d \geq 3 \): transient.