01/13

4.4 Limiting Probabilities.

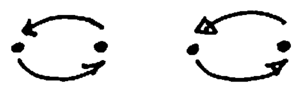
(Q). Consider a Markov chain. Let the process run for a long time, starting in some state i. ~~Does~~ Will it "mix" well? i.e. does

$$\lim_{n\to\infty} \mathbb{P}\{X_n = j \mid X_0 = i\} \qquad (*)$$

exist, and is the limit independent of starting state i?

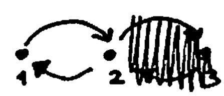
Sometimes NOT. Obstructions:

(a) Reducible M. chains:



This M. chain consists of 2 (non-communicating) classes, thus limiting probabilities $(*)$ must depend on the starting state!

(b) Periodic M. chains



Starting in state 1, process will visit state 2 on even times only.
Thus, ~~no limiting prob limit~~ probabilities

$$\mathbb{P}\{X_n = 2 \mid X_0 = 1\} = \{0, 1, 0, 1, 0, 1, \dots\}$$

No limit.

However: these are the only obstructions.
(Ergodic Thm for M. chains) no state has period

Thm Consider a finite, irreducible, aperiodic M. chain.
Then, $\forall i, j$, $\pi_j := \lim_{n\to\infty} \mathbb{P}\{X_n = j \mid X_0 = i\}$
exists and is independent of i. (W/o Proof - see Walsh)

These probabilities π_j are called "stationary distribution" on the M. chain.

Q: How to compute π_i in terms of transition prob's P_{ij}?

Answer:

THM The limiting probabilities π_i must satisfy:

$$\pi_j = \sum_i \pi_i P_{ij}, \quad \sum_j \pi_j = 1. \qquad (*)$$

- By law of Total Prob,

$\cancel{P\{X_{n+1}=j\}\, X_n}$

$$P\{X_{n+1}=j\} = \sum_i P\{X_{n+1}=j \mid X_n=i\} \cdot P\{X_n=i\}$$

Take limit as $n\to\infty$ $\Rightarrow$

$$\pi_j = \sum_i P_{ij}\pi_i . \quad \Rightarrow \text{first eq.}$$

- $\sum_j P\{X_n=j\} = 1.$

Take limit as $n\to\infty$ $\Rightarrow$

$$\sum_j \pi_j = 1 \quad \Rightarrow \text{second eq.}$$

THM In a finite, irreducible, aperiodic M. chain, the solution $\pi=(\pi_1,\dots,\pi_n)$ to $(*)$ is unique.
(w/o proof - see Welsch)

Remark In matrix form, $\pi = \begin{pmatrix}\pi_1\\ \vdots \\ \pi_n\end{pmatrix}$ $\Rightarrow$

$$\boxed{\pi^T = \pi^T P}$$

$\Leftrightarrow \quad \pi = P^T\pi.$

Hence: π is an eigenvector of P^T corresponding to eigenvalue 1.

Ex Every minute, a server is busy or idle

If busy, it will ~~be busy with prob 0.~~ become idle next minute with prob. 0.7

If idle, it becomes busy next min with prob. 0.1.

What % of time is the server busy?

~~State~~ 1 = "busy", 2 = "idle"

$$P = \begin{bmatrix} 0.3 & 0.7 \\ 0.1 & 0.9 \end{bmatrix}$$

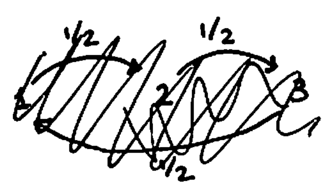
$$[\pi_1 \; \pi_2] = [\pi_1 \quad \pi_2] \begin{bmatrix} 0.3 & 0.7 \\ 0.1 & 0.9 \end{bmatrix}$$

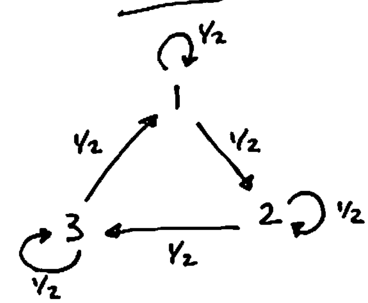
$$\begin{cases} \pi_1 = 0.3\pi_1 + 0.1\pi_2 \\ \pi_2 = 0.7\pi_1 + 0.9\pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

Solving gives $\pi_1 = \frac{1}{8} = 0.125$, $\pi_2 = \frac{7}{8} = 0.875$.

Thus, server is busy 12.5% of the time.

Ex Markov Chain :





$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

Solving $\pi = \pi P$ gives

$\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$. (Also clear by symmetry).

~~Ex Symmetric random walk; no limiting stationary distribution! (If there were one,~~

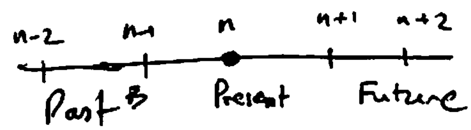
4.8. Time reversal.

Run Markov chain backwards? Yes; it is ~~Thi~~ still a M. chain:

Prop (Time reversal) If $X_0, X_1, X_2, \dots$ is a M. chain then $X_n, X_{n-1}, X_{n-2}, \dots$ is a Markov chain, too.

Since $X_0, X_1, X_2, \dots$ is a M. chain, (conditional) distribution of future

$$(X_{n+1}, X_{n+2}, \dots \mid X_n)$$



is independent of the past: $X_0, X_1, \dots, X_{n-1}$.

Thus, (conditional) distr. of the past

$$(X_0, X_1, \dots, X_{n-1} \mid X_n)$$

is independent of the future: $X_{n+1}, X_{n+2}, \dots$ $\Rightarrow$ QED

(Here we use symmetry: that if A is indep from B then B is indep from A)

Q: Transition probabilities ~~of the re~~ Q_{ij} of the reversed chain?

Let's assume n is large to eliminate any influence of the initial state X_0.

$$Q_{ij} = \mathbb{P}\{X_n = j \mid X_{n+1} = i\}$$

$$= \frac{\mathbb{P}\{X_n = j,\ X_{n+1} = i\}}{\mathbb{P}\{X_{n+1} = j\}}$$

$$= \frac{\mathbb{P}\{X_{n+1} = i \mid X_n = j\} \cdot \mathbb{P}\{X_n = j\}}{\mathbb{P}\{X_{n+1} = i\}} \qquad (\sim \text{Bayes formula})$$

$$\xrightarrow{n \to \infty} \frac{\pi_j P_{ji}}{\pi_i}$$

We proved:

THM The transition probabilities Q_{ij} in the reverse M. chain are

$$Q_{ij} = \frac{\pi_j P_{ji}}{\pi_i}$$

where P_{ij} are trans. prob's of the original (forward) chain, and π_i are the limiting prob's of that chain.

Ex: Check that the backwards M. chain has the same limiting prob's π_i.

Def A Markov chain is called <u>reversible</u> if $Q_{ij} = P_{ij}$ $\forall i,j$.
i.e. the backward & forward chains have same distr.

Ex (a) Symmetric random walk is reversible;
(b) Non-symmetric (with $p \neq \frac{1}{2}$) is NOT

THM (<u>Criterion</u> of <u>reversibility</u>) Consider a finite, irreducible, aperiodic M. chain.

(i) Suppose the chain is reversible. If π_i are the limiting probabilities, then
$$\pi_i P_{ij} = \pi_j P_{ji} \quad \forall i,j.$$

(ii) Vice versa, suppose there exists numbers $\pi_i \geq 0$, $\sum_i \pi_i = 1$, such that
$$\pi_i P_{ij} = \pi_j P_{ji} \quad \forall i,j. \qquad (*)$$
Then π_i are the limiting probabilities, and the M. chain is reversible.

(i) follows from Thm p.126:
$$P_{ij} = Q_{ij} = \frac{\pi_i P_{ij}}{\pi_j}.$$

(ii) ~~First check that~~ To check that π_i are limiting prob's, by Thm p.124 it is enough to check that
$$\sum_i \pi_i P_{ij} = \pi_j.$$
$$\sum_i \pi_i P_{ij} \overset{(*)}{=} \sum_i \pi_j P_{ji} = \pi_j \underbrace{\sum_i P_{ji}}_{=1} = \pi_j. \qquad \text{Done!}$$

The M. chain is reversible; by Thm p.126, $Q_{ij} = P_{ij}$ as in (i).