4.4 Limiting Probabilities.

Consider a Markov chain. Let the process run for a long time, starting in some state \(i\). Does it "mix" well? i.e., does

\[
\lim_{n \to \infty} P \{ X_n = j \mid X_0 = i \} \tag{X}
\]

exist, and is the limit independent of starting state \(i\)?

Sometimes not. Obstructions:

(a) **Reducible M. chains**

\[\begin{array}{c}
\circlearrowright \\
\circlearrowright \\
\end{array}\]

This M. chain consists of 2 (non-communicating) classes, thus limiting probabilities (X) must depend on the starting state!

(b) **Periodic M. chains**

\[\begin{array}{c}
\circlearrowright \\
\circlearrowright \\
\end{array}\]

Starting in state 1, process will visit state 2 on even times only. Thus, no limiting prob-limit probabilities

\[P \{ X_n = 2 \mid X_0 = 1 \} = \{0, 1, 0, 1, 0, 1, \ldots\}\]

No limit.

However: there are the only obstructions.

(Ergodic than for M. chains) no state has period

Thus, consider a finite, irreducible, aperiodic M. chain.

Then, \(\pi_j\), \(\pi_j = \lim_{n \to \infty} P \{ X_n = j \mid X_0 = i \}\)

exists and is independent of \(i\). (W/o Proof - see Walsh)

These probabilities \(\pi_j\) are called "stationary distribution" on the Mchain.
Q: How to compute $\pi_i$ in terms of transition prob's $P_{ij}$?

Answer:

THM: The limiting probabilities $\pi_i$ must satisfy:

$$\pi_j = \sum_i \pi_i P_{ij}, \quad \sum_j \pi_j = 1. \quad (*)$$

By law of Total Prob,

$$P_{X_{n+1}=j} X_n = \sum_i P_{X_{n+1}=j | X_n=i} \cdot P_{X_n=i}$$

Take limit as $n \to \infty$ \Rightarrow

$$\pi_j = \sum_i P_{ij} \pi_i \Rightarrow \text{Rrst eq.}$$

$$\sum_i P_{ij} = 1 \Rightarrow \text{Second eq.}$$

THM: In a finite, irreducible, aperiodic M. chain, the solution $\pi = (\pi_1, \ldots, \pi_n)$ to $(*)$ is unique (w/o proof-see Welsh).

Remark: In matrix form, $\pi = \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_n \end{pmatrix}$ \Rightarrow

$$\pi^T = \pi^T P$$

$$\Rightarrow \pi = P^T \pi$$

Hence: $\pi$ is an eigenvector of $P^T$ corresponding to eigenvalue 1.
Every minute, a server is busy or idle. If busy, it will become idle next minute with prob. 0.7. If idle, it becomes busy next min with prob. 0.1. What % of time is the server busy?

\[ p = \begin{bmatrix} 0.3 & 0.7 \\ 0.1 & 0.9 \end{bmatrix} \]

\[ [\pi_1, \pi_2] = \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 \\ 0.1 & 0.9 \end{bmatrix} \]

\[ \begin{cases} \pi_1 = 0.3\pi_1 + 0.1\pi_2 \\ \pi_2 = 0.7\pi_1 + 0.9\pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases} \]

Solving gives \( \pi_1 = \frac{1}{8} = 0.125 \), \( \pi_2 = \frac{7}{8} = 0.875 \). Thus, server is busy \( 12.5\% \) of the time.

Markov Chain:

The steady-state probabilities are given by the limiting stationary distribution.

\[ \pi_1 = \pi_2 = \pi_3 = \frac{1}{3} \]

(Also clear by symmetry.)

Symmetric random walk: no limiting stationary distribution.
4.8. Time reversal.

Run Markov chain backwards? Yes, it is.

Prop (Time reversal). If \( X_0, X_1, X_2, \ldots \) is a M. chain
then \( X_n, X_{n-1}, X_{n-2}, \ldots \) is a Markov chain, too.

Since \( X_0, X_1, X_2, \ldots \) is a M. chain,
the (conditional) distribution of future
\[(X_{n+2}, X_{n+1}, \ldots | X_n)\]
is independent of the past, \( X_0, X_1, \ldots, X_{n-1} \).

Thus, (conditional) distr. of the past
\[(X_0, X_1, \ldots, X_{n-1} | X_n)\]
is independent of the future: \( X_{n+1}, X_{n+2}, \ldots \rightarrow \text{OED} \).

Q: Transition probabilities of the – Qij of the reversed chain?

Let's assume \( n \) is large to eliminate any influence of the initial state \( X_0 \).

\[ Q_{ij} = P\{X_n = j | X_{n+1} = i\} \]
\[ = \frac{P\{X_n = j, X_{n+1} = i\}}{P\{X_{n+1} = i\}} \]
\[ = \frac{P\{X_n = j\} \cdot P\{X_{n+1} = i\}}{P\{X_{n+1} = i\}} \]
\[ \stackrel{\text{n} \rightarrow \infty}{\rightarrow} \frac{\pi_j P_{ij}}{\pi_i} \quad \text{We proved.} \]

THM The transition probabilities \( Q_{ij} \) in the reverse M. chain are

\[ Q_{ij} = \frac{\pi_j P_{ij}}{\pi_i} \]

where \( P_{ij} \) are trans. probs. of the original (forward) chain,
and \( \pi_i \) are the limiting probs. of that chain.

Ex: Check that the backwards M. chain has the same limiting probs. \( \pi_i \).
Def: A Markov chain is called **reversible** if $q_{ij} = p_{ij}$ for all $i, j$.

Ex. (a) Symmetric random walk is reversible.
(b) Non-symmetric (with $p 
eq \frac{1}{2}$) is not.

**THM** (Criterion of reversibility) Consider a finite, irreducible, aperiodic M. chain. Suppose the chain is reversible.

(i) If $\pi$ are the limiting probabilities, then
$$\pi_i p_{ij} = \pi_j p_{ji} \quad \forall i, j.$$ 

(ii) Vice versa, suppose there exists numbers $\pi_i \geq 0$, $\sum_i \pi_i = 1$, such that
$$\pi_i p_{ij} = \pi_j p_{ij} \quad \forall i, j.$$ (i)

Then $\pi_i$ are the limiting probabilities and the M. chain is reversible.

(i) follows from Thm p.126:
$$\pi_j = q_{ij} = \frac{\pi_i p_{ij}}{\pi_j}.$$ 

(ii) First check that $\pi$ are limiting prob's, by Thm p.124 it is enough to check that
$$\sum_i \pi_i p_{ij} = \pi_j.$$ 

$$\sum_i \pi_i p_{ij} = \sum_i \pi_j p_{ji} = \pi_j \sum_i p_{ji} = \pi_j.$$ Done!

The M. chain is reversible. by Thm p.126, $q_{ij} = p_{ij}$ as in (i).