Homework 6: Additional Problems **Math 525, Winter 2017**

1. Let X be a random variable which takes values $0, 1, 2, \ldots, n$ and satisfies $\mathbb{E}[X] = \text{Var}(X) = 1$. Show that, for every positive integer k,

$$\mathbb{P}(X > k) \leq \frac{1}{k^2}.$$

2. Consider independent random variables $X, Y \sim \text{Unif}\,[0, 1]$. Let $Z = X + Y$ and $W = X - Y$. Compute the following correlations:

(a) $\rho_{X,Y}$

(b) $\rho_{X,Z}$

(c) $\rho_{Z,W}$.

3. Particles are subject to collisions, which cause them to split into two parts with each part a fraction of the parent. Suppose that this fraction is uniformly distributed between 0 and 1, for each collision. Following a single particle through several splittings, we obtain a fraction of the original particle

$$Z = X_1 \cdot X_2 \cdots X_n,$$

with $X_j \sim \text{Unif}\,[0, 1]$ being independent. Show that the density of the random variable Z_n is

$$f(z) = \frac{1}{(n-1)!}(-\log z)^{n-1}.$$

Hint: Show that $Y_k = -\log X_k$ is exponentially distributed. Use this to find the density of $S_n = Y_1 + Y_2 + \cdots + Y_n$, and from this the CDF and density of Z_n.