Midterm Exam 1

SOLUTIONS

Math 130A, Prof. Roman Vershynin
Fall 2017 **Name:** ______________________

Read the following information before starting the exam.

- No calculators or textbooks are allowed on this exam.
- Show all work, clearly and in order, if you want to get full credit. Points may be taken off if it is not clear how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. Points may be taken off for rambling and for incorrect or irrelevant statements.

1. (*10 points*) Suppose events E and F are mutually exclusive with $P(E) = 0.3$ and $P(F) = 0.5$. Compute ~~$P(E^c \cap F^c)$~~. $P(E^c \cap F^c)$

$$P(E^c \cap F^c) = P((E \cup F)^c) \quad \text{(by de Morgan's Law)}$$

$$= 1 - P(E \cup F)$$

$$= 1 - P(E) - P(F) \quad \text{(since A, B are mutually exclusive)}$$

$$= 1 - 0.3 - 0.5 = 0.2$$

2. (*10 points*) Alice and Bob each randomly picks an integer number between 1 and n. Assume that all possible combinations of two numbers are equally likely to be picked. What is the probability that Alice's number is bigger than Bob's?

Solution 1 : Consider the events

$A = \{$Alice's number is bigger$\}$,
$B = \{$Bob's number is bigger$\}$,
$E = \{$the numbers are equal$\}$.

Then $P(A) + P(B) + P(E) = 1$,

$P(A) = P(B)$ by symmetry,

and $P(E) = \frac{n}{n^2} = \frac{1}{n}$ (since n = # ways to choose the same number, n^2 = # ways to choose any numbers)

From this we get $P(A) = \frac{1 - P(E)}{2} = \frac{1 - 1/n}{2} = \frac{n-1}{2n}$.

Solution 2 :

Total # ways to choose 2 numbers is n^2.

ways to choose 2 numbers such that the first is bigger is $\binom{n}{2}$
(it's the same as # of unordered pairs).

Thus the probability in question equals

$$\frac{\binom{n}{2}}{n^2} = \frac{n(n-1)}{2n^2} = \frac{n-1}{2n}.$$

3. (*10 points*) I have a fair coin and a two-headed coin. I choose one of the two coins randomly with equal probability and flip it. It comes up heads. What is the probability that I flipped the two-headed coin?

Consider the events

$H = \{\text{coin comes up heads}\}$,

$T = \{\text{coin is two-headed}\}$, $F = T^c = \{\text{coin is fair}\}$

Bayes Formula gives

$$P(T \mid H) = \frac{P(H \mid T)\, P(T)}{P(H \mid T)\, P(T) + P(H \mid F)\, P(F)}$$

$$= \frac{1 \cdot \frac{1}{2}}{1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}} = \frac{2}{3}$$

4. (*10 points*) In a group of n people, a person tells the news to a second person, who in turn tells it to a third person, and so on. Suppose that at each step, a person tells the news to just one of the other $n-1$ people chosen at random. Compute the probability that the news it told 20 times without being repeated to any person.

- The total number of ways to choose 20 people who will hear the news is $(n-1)^{20}$ (they may repeat)
- The number of ways to choose all these 20 people so that no one will repeat is $(n-1)(n-2)(n-3)\cdots(n-20)$.

Thus, the probability in question equals

$$\frac{(n-1)(n-2)(n-3)\cdots(n-20)}{(n-1)^{20}} = \frac{(n-1)!}{(n-21)!(n-1)^{20}}.$$

5. (*10 points*) A certain hedge fund reports that among its 100 clients, there are 70 professionals, 60 married persons, 50 California residents, 40 married professionals, 30 California professionals, 20 married California residents, and 15 married California professionals. Prove that this report contains mistakes.

Choose a client randomly; consider the events

$F = \{\text{client is a professional}\}$,

$M = \{\text{client is married}\}$,

$C = \{\text{client is a CA resident}\}$.

By Inclusion - Exclusion Principle,

$$P(F \cup M \cup C) = P(F) + P(M) + P(C) - P(F \cap M) - P(F \cap C) - P(M \cap C) + P(F \cap M \cap C)$$

$$= 0.7 + 0.6 + 0.5 - 0.4 - 0.3 - 0.2 + 0.15$$

$$= 1.05$$

Probability can never be greater than 1. Therefore, the report contains mistakes.

6. (*10 points*) Lung cancer is twice as likely to develop when a person lives in a polluted city as it is when they live in a clean city. 20% of world's population lives in polluted cities. What percent of people having lung cancer live in polluted cities?

Choose a person randomly; consider the events

$B = \{\text{person lives in a polluted city}\}$ ("Good")

$G = \{\text{person lives in a clean city}\}$ ("Bad")

$C = \{\text{person has lung cancer}\}$.

We are given that

$$P(C|B) = 2P(C|G), \qquad P(B) = 0.2. \qquad (*)$$

Bayes formula gives

$$P(B|C) = \frac{P(C|B)P(C)}{P(C|B)P(B) + P(C|G)P(G)}$$

Denote $p = P(C|G)$; then

$$P(B|C) = \frac{2p \cdot 0.2}{2p \cdot 0.2 + p \cdot 0.8} \qquad \text{(using } (*)\text{)}$$

$$= \frac{2 \cdot 0.2}{2 \cdot 0.2 + 0.8} = \frac{1}{3}.$$

<u>Answer</u>. $\simeq 33\%$

(Total number of points in this exam = 60)