Midterm Exam 2

SOLUTIONS

Math 130A, Prof. Roman Vershynin
Fall 2017 **Name:** ______________________

Read the following information before starting the exam.

- No calculators or textbooks are allowed on this exam.
- Show all work, clearly and in order, if you want to get full credit. Points may be taken off if it is not clear how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. Points may be taken off for rambling and for incorrect or irrelevant statements.

1. (*10 points*) Toss of a single die, and consider the following events. A is the event that an odd number is observed; B is the event that an even number is observed; C is the event that the top face of the die has either one or two dots.

a. (*5 pts*) Are A and B independent events? Prove or disprove.

$P(A) = P(B) = \frac{1}{2}$. But $P(A \cap B) = 0$, since A, B are mutually exclusive.

Hence $P(A \cap B) \neq P(A) \cdot P(B)$

and A, B are not independent.

b. (*5 pts*) Are A and C independent events? Prove or disprove.

$P(C) = \frac{2}{6} = \frac{1}{3}$,

$A \cap C = \{\text{one dot}\} \Rightarrow P(A \cap C) = \frac{1}{6}$

$P(A)\,P(C) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$, too

Hence A, C are independent

2. (*10 points*) A page of the book contains, on average, two typos. Suppose you open the book and spot a typo on the page. Estimate the probability that this page has at least one more typo.

Use the Poisson distribution.

Let $X = \#$ typos on the page. Then $X \sim \text{Poisson}(2)$.

$$\mathbb{P}\{X \geq 2 | X \geq 1\} = \frac{P\{X \geq 2\}}{P\{X \geq 1\}}$$

$$= \frac{1 - p(0) - p(1)}{1 - p(0)} \quad \text{(where } p(x) \text{ denotes the pmf of } X\text{)}$$

$$= \frac{1 - e^{-2} - 2e^{-2}}{1 - e^{-2}} = \frac{1 - 3e^{-2}}{1 - e^{-2}}.$$

3. (*10 points*) Let X be a random variable whose cdf (cumulative distribution function) is given by

$$F(x) = \begin{cases} 0, & x < -1 \\ 1/4, & -1 \le x < 0 \\ 3/4, & 0 \le x < 1 \\ 1, & x \ge 1. \end{cases}$$

a. (*5 pts*) Compute the pmf (probability mass function) of X.

$$p(-1) = \frac{1}{4} - 0 = \frac{1}{4}$$

$$p(0) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

$$p(1) = 1 - \frac{3}{4} = \frac{1}{4}.$$

b. (*5 pts*) Compute $E[X]$.

$$E[X] = -1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} = 0.$$

4. (*10 points*) An exam consists of two rounds, with two questions in each round. To pass the exam, the student needs to answer at least one question correctly in each round. Alice estimates her probability to answer a question correctly to be p, independently for each question. Compute the probability that Alice will pass the exam.

$P(\text{pass first round}) = P(E_1 \cup E_2)$ where $E_i = \{\text{answer } i \text{ is correct}\}$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$ (i.-e.p.)

$= p + p - p^2$ (by independence).

$= 2p - p^2.$

$P(\text{pass second round}) = 2p - p^2$ (by the same reasoning)

$P(\text{pass exam}) = P(\text{pass both rounds}) = (2p-p^2)^2$ (by independence).

5. (*10 points*) N balls are put into M bins at random (i.e. each ball is put independently in a randomly chosen bin). Compute the expected number of empty bins.

Hint: express the number of empty boxes in terms of indicators – random variables X_i that take value 1 if i-th bin is empty and 0 otherwise.

Let $X = \#$ empty bins; then $X = \sum_{i=1}^{M} X_i$ and

$$E[X] = \sum_{i=1}^{M} E[X_i]$$

$$E[X_i] = P\{X_i = 1\} = P\{\text{bin } i \text{ is empty}\}$$

$$= P\{\text{none of the } N \text{ balls are put in bin } i\}$$

$$= \left(1 - \frac{1}{M}\right)^N \quad \text{by independence. } \left(\frac{1}{M} \text{ is the prob. to put a ball in bin } i\right)$$

Hence $E[X] = M\left(1 - \frac{1}{M}\right)^N$.

6. (*10 points*) A 40-year old person dies during the next year of his life with probability p. An insurance company has N policyholders who just turned 40. Each of them pays the company the premium of r dollars for the next year. If a policyholder dies during that year, the company pays the policyholder s dollars.

a. (*5 pts*) Compute the expected profit of the company for the next year. (The profit equals the total premium minus total payment.)

Let $X :=$ # policyholders who die in the next year; $X \sim \text{Binom}(N, p)$.

Total premium $= rN$, total payment $= sX$, profit $= rN - sX$.

Expected profit $= E[rN - sX] = rN - s\,E[X]$

$= rN - spN$ (since $X \sim \text{Binom}(N,p)$)

$= (r - sp)N$.

b. (*5 pts*) Compute the variance of the company's profit for the next year.

$\text{Var}(rN - sX) = s^2 \text{Var}(X) = s^2 N p(1-p)$

(since $X \sim \text{Binom}(N,p)$)

(Total number of points in this exam = 60)