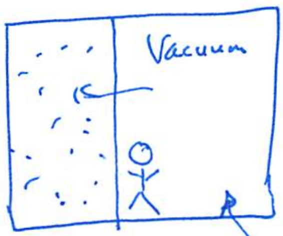
2 Microscopic chaos — macroscopic order

Ex:



Room

Due to diffusion $\forall$ oxygen molecule can be ~~somewhere~~ anywhere in the room.

~~fraction oxygen molecules~~

LLN: fraction of oxygen molecules here ~~is proportional to $\frac{1}{2}$~~ is $\simeq \frac{1}{2}$ with prob. (almost) 1.

$\exists$ chance all molecules leave, create vacuum!

$\Rightarrow$ Stat. Mechanics

3 Monte-Carlo method.

- Numeric integration $\int_0^1 g(x)\,dx = ?$

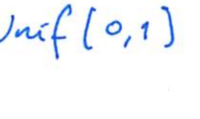
Interpret $\Updownarrow$ $\quad \| \quad E[g(X)]$ where $X \sim \mathrm{Unif}[0,1]$

let $X_1, \dots, X_n \sim \mathrm{Unif}[0,1]$ iid

LLN $\Rightarrow$ $$\boxed{\frac{g(X_1)+\dots+g(X_n)}{n} \to E[g(X)] = \int_0^1 g(x)\,dx.}$$

Computable! (Randomized algorithm)

- Very flexible: $\forall$ domain $D \subset \mathbb{R}^n$ can numerically compute $$\int_D g(x)\,dx$$



by sampling $X_i \sim \mathrm{Unif}(D)$. Good approximation regardless of dimension n, how complex D is.

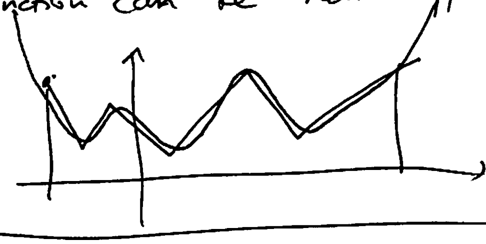
- Manhattan project.

Application of LLN for Approximation Theory

Weierstrass Approximation Theorem:

- "$\forall$ continuous function on a closed bounded interval can be (uniformly) approximated by a polynomial"

- Remark: function can be non-differentiable $\Rightarrow$ Taylor won't work



$$\boxed{\begin{array}{l}\textbf{THM} \quad \forall\, g : [0,1] \to \mathbb{R} \text{ continuous} \\ \exists \text{ sequence of polynomials } p_n(x) \text{ s.t. } \quad \forall x \in [0,1]: \\ \qquad p_n(x) \to g(x) \quad \text{as } n \to \infty \end{array}}$$

Heuristic proof: • Fix $x \in [0,1]$, define

$$X_1, X_2, \dots, X_n \sim Ber(x)$$

$$S_n := X_1 + \dots + X_n \sim Binom(n, x)$$

• LLN: with probability 1,

$$\frac{S_n}{n} \to x \quad \text{as } n \to \infty.$$

$$\Rightarrow g\left(\frac{S_n}{n}\right) \to g(x) \quad \text{as } n \to \infty \qquad \text{(continuity)}$$

$$\Rightarrow \underbrace{E\left[g\left(\frac{S_n}{n}\right)\right]}_{\parallel\ S_n \sim Binom(n,x)} \to g(x).$$

$$\boxed{\sum_{k=0}^{n} g\left(\frac{k}{n}\right) x^k (1-x)^{n-k}} := p_n(x) \quad \text{polynomial} \qquad \square$$

Remark: $p_n(x)$ called Bernstein polynomial.

- simple, constructive approximation!
- actually, convergence is uniform: $\max_{x \in [0,1]} |p_n(x) - g(x)| \to 0$ as $n \to \infty$.

WHAT IS PROBABILITY?

- Frequentist ~~so~~ vs Bayesian approaches

prob is revealed by ~~perform~~ repeating experiment, ~~frequency $\overset{LLN}{=}$ pro~~

prob $\approx$ frequency (LLN)

prob = measure of our belief, constantly adjusted in light of new information

- Prob (Rain tomorrow) = ? Frequentist — does not make sense.

Bayesian: measure of ~~belief~~ our belief, based on past ~~exp obs.~~ past experience &

~~We Fac~~ Adjusting based on: Sunny today. P(Rain tomorrow) ~~= pri~~

prior = P(Rain tomorrow)

Posterior = P(Rain tom | Sun today)

$$\underset{\text{Bayes}}{=} \frac{P(\text{Sun today} \mid \text{Rain tomorrow})\, P(\text{Rain tom})}{\underbrace{P(\text{Sun today} \mid \text{Rain tomorrow})}\, P(\text{Rain tom}) + \cancel{P(\text{Rain today} \mid \text{Rain tod})\, P(\text{Rain Tod})}\ P(\text{Sun today} \mid \text{Sun tom})\, P(\text{Sun tom})}$$

Adjustments based on Bayes formula