1. Brownian Motion

- Jittery movement of micro-particles (e.g. dust) in water.
- Observed by Greeks, described by R. Brown (1827)
- Explained by A. Einstein (1905): Brownian motion is caused by the collision of the particles by fast-moving molecules in water molecules.
- Confirmed existence of molecules that matter that matter is made of: particles (atoms, molecules)
- Allowed to compute size/mass

- Mathematical Model of B.M

  (simplified, and in 1D)

  Limit of a random walk when steps size \( \to 0 \):

  \[
  \begin{array}{cccc}
  0 & \delta & 2\delta & \ldots & t = n\delta \\
  \end{array}
  \]

  Divide time interval \([0, t]\) into sub-intervals of length \( \delta \).
  Each step is right (prob = \( \frac{1}{2} \)) or left (prob = \( \frac{1}{2} \)) step size = \( \sqrt{\delta} \).

  After time \( t \), particle is at

  \[
  W_t^{(n)} = \sum_{i=1}^{n} X_i, \quad X_i = \{ \sqrt{\delta}, \text{prob } \frac{1}{2} \}
  \]

  \[
  E[W_t^{(n)}] = \sum_{i=1}^{n} E[X_i] = 0
  \]

  \[
  \text{Var}(W_t^{(n)}) = \sum_{i=1}^{n} \text{Var}(X_i) = n\delta = t.
  \]

  \[
  \text{CLT } \Rightarrow \text{ as } n \to \infty,
  \]

  \[
  \frac{W_t}{\sqrt{t}} \to N(0, 1).
  \]
Hence, a characterization of Brownian motion \((B_t)_{t \geq 0}\):

- \(B_0 = 0\)
- \(B_s - B_t \sim N(0, t - s)\) \(\forall 0 \leq s \leq t\)
- \(\forall t_1 < t_2 \ldots < t_n\), the r.v.'s \(W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}\) are independent ("independent increments")