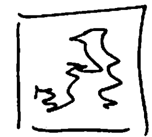
Implications of CLT.

1. Brownian Motion



- Jittery movement of micro-particles (e.g. dust) in water.
- Observed ~~by~~ since Greeks, described by R. Brown (1827)
- Explained by A. Einstein (1905):
  B.M. is caused by the collision of the particles by fast-moving ~~molecules in~~ water molecules.
- Confirmed ~~existence of molecules = that matter~~ ~~that~~ matter is made of particles (atoms, molecules)
- Allowed to compute size/mass of molecules

4 Einstein papers in 1905:
1. $E = mc^2$ paper
2. Special relativity
3. Photoelectric effect
4. Brownian motion

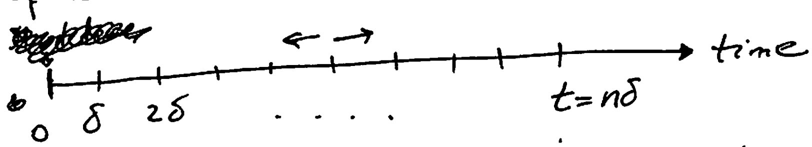
"ANNUS MIRABILIS"

Previously — this hypothesis was not commonly accepted, e.g. Mach, Ostwald

- Mathematical Model of B.M

(simplified, and in 1D)

Limit of a random walk when steps size $\to 0$:



Divide time interval $[0, t)$ into n tiny sub-intervals of length δ.

Each step is Right (prob $= \frac{1}{2}$) or Left (prob. $1/2$) step size $= \sqrt{\delta}$.

After time t, particle is at

$$W_t = \sum_{i=1}^{n} X_i, \qquad X_i = \begin{cases} \sqrt{\delta}, & \text{prob } 1/2 \\ -\sqrt{\delta}, & \text{prob } 1/2. \end{cases}$$

$$E[W_t] = \sum_{i=1}^{n} E[X_i] = 0$$

$$Var(W_t) = \sum_{i=1}^{n} Var(X_i) = n\delta = t.$$

CLT $\Rightarrow$ as $n \to \infty$,

$$\frac{W_t - 0}{\sqrt{t}} \to N(0,1). \quad \Rightarrow \quad \boxed{W_t \sim N(0, t).}$$

Normal!

~~Hence~~

~~Hence:~~ ~~B~~ CHARACTERIZATION of

~~Summary~~ ~~of~~ Brownian motion: $(B_t)_{t \geq 0}$:

- $B_0 = 0$
- $B_t - B_s \sim N(0, t-s) \quad \forall\, s \leq t$
- $\forall\, t_1 < t_2 < \dots < t_n$, the r.v's

  $W_{t_1}, \; W_{t_2} - W_{t_1}, \; \dots, \; W_{t_n} - W_{t_{n-1}}$

  are independent

  ("Independent increments")