Characteristic functions

Def Let X be a r.v. The char. function of X is
$$\phi(t) := \mathbb{E}[e^{itX}] = \underset{\substack{\uparrow \\ \text{MGF}}}{M}(it), \quad t \in \mathbb{R}$$

Advantage over MGF: char. function is defined $\forall t \in \mathbb{R}$, since $|e^{itx}| = 1$.

Ex (a) $X \sim N(0,1) \Rightarrow \phi(t) = e^{-t^2/2}$

(b) $X = \pm 1$ with prob $\frac{1}{2}$ each $\Rightarrow \phi(t) = \cos(t)$ (Check!)

(c) $X \sim$ cauchy, i.e. pdf $f(x) = \frac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$.

$\Rightarrow \quad \phi(t) = e^{-|t|}$ ~~(can be checked using Fourier inv~~

(recall MGF of X is $= \infty$).

Char. functions share many properties with MGF, including:
$$\phi_{\Sigma X_i}(t) = \phi_{X_1}(t) \cdots \phi_{X_n}(t)$$
if X_i are independent.

~~Rem Remark If X has dens~~

Remark Char. function also determines the distr. of X uniquely.

Remark. Let X have density $f(x)$. Then
$$\phi(x) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx = \hat{f}(-t), \quad \text{Fourier transform of } f.$$

Fourier inversion formula:
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t)\,dt$$
allows to compute density from char. function.

Prop Let $X_1, \ldots, X_n \sim$ Cauchy, independent. Then
$$X := \frac{X_1 + \cdots + X_n}{n} \sim \text{Cauchy}.$$

$\ulcorner$ $\phi_X(t) = \prod_i \phi_{X_i/n}(t)$ ~~$= E\phi_{X_i}(t/n) =$~~ (by indep)

$$= \prod_i \phi_{X_i}(t/n) = \prod_i e^{-|t/n|} \quad \text{(see p. 83)}$$

$$= e^{-|t|}$$ ← char. function of Cauchy. Uniqueness ⇒ QED $\lrcorner$

Remark Won't contradict CLT (~~no~~ variance of Cauchy $= \infty$).