Density (PDF) of multivariate normal dist:

- Standard normal:
  \[ Z = (Z_1, \ldots, Z_n) \sim N(0, I_n) : \]
  \[
  f_Z(z) = f_{Z_1}(z_1) \cdots f_{Z_n}(z_n) \quad \text{(indep.)}
  \]
  where \(|Z|\) is the length of \(Z\).

\[
  f_{Z_i}(z_i) = \frac{1}{(2\pi)^{n/2}} e^{-z_i^2/2}
\]

- General normal:
  \[ X = \mu + A Z, \quad X \sim N(\mu, \Sigma) \]

  \[ X = \mu + A Z. \]

  \[ E(X) = \mu; \]


  \[ J = \det(A) = \sqrt{\det(AA^T)} = \sqrt{\det(\Sigma)}, \]

  \[ f_X(x) = \frac{1}{(|J|)^{n/2}} e^{-x^T \Sigma^{-1} x} \]

  \[ f_Z(z) = \frac{1}{(2\pi)^{n/2}} e^{-z^T z / 2}, \]

  where \( J \) is the Jacobian matrix.
$z^2 = z^T z = (A^T x)^T (A^T x)$

$z = A^T (x - \mu)$, so

$|z|^2 = z^T z = (A^T (x - \mu))^T (A^T (x - \mu))$

$= (x - \mu)^T (A^T)^T A^{-1} (x - \mu)$

$= \xi^T \Sigma^{-1} \xi$

$(A^T A)^{-1} = \Sigma^{-1}$.

Hence:

$f_{X}(x) = \frac{1}{(2\pi)^{n/2}} \exp \left( -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right) \cdot \frac{1}{\sqrt{\det(\Sigma)}}$

⇒ we proved:

**THE PDF of**

\[ f(x) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp \left[ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right], \quad x \in \mathbb{R}^n. \]

- Compare to univariate $X \sim N(\mu, \sigma^2)$:
  \[ f(x) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp \left( -\frac{1}{2} (x - \mu)^2 / \sigma^2 \right) \]

- Linear transformation of a standard normal PDF

\[ N(0, I) \rightarrow N(\mu, \Sigma) \]

level sets
Ex. \[ E(X_1) = E(X_2) = 0, \]
\[ \text{var}(X_1) = 1, \quad \text{var}(X_2) = 4, \]
\[ \sigma_{X_1,X_2} = \sigma \frac{1}{2}, \]
\[ \frac{\text{Cov}(X_1,X_2)}{\sqrt{\text{var}(X_1) \text{var}(X_2)}} = \frac{\text{Cov}(X_1,X_2)}{\sqrt{1 \cdot 4}} = \frac{1}{2}. \]

Compute PDF. \[ \text{Cov}(X_1,X_2) = 1. \]

\[ \Sigma = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}, \quad \Sigma^{-1} = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}, \]

\[ \det(\Sigma) = 3. \]

\[ f(x) = \frac{1}{2\pi\sqrt{3}} \exp\left( -\frac{1}{2} x^T \Sigma^{-1} x \right) \]

\[ = \frac{1}{2\pi\sqrt{3}} \exp\left( -\frac{1}{2} \left( \frac{4}{3} x_1^2 + \frac{1}{3} x_2^2 - \frac{2}{3} x_1 x_2 \right) \right) \]

\[ = \frac{1}{2\pi\sqrt{3}} \exp\left( -\frac{4x_1^2 + x_2^2 - 2x_1 x_2}{6} \right) \]

Eigenvalues / eigenvectors: \[ \lambda_1 = 4.3, \quad \mathbf{v}_1 = \begin{bmatrix} 0.3 \\ 1 \end{bmatrix} \]
\[ \lambda_2 = 0.7, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 3 \end{bmatrix} \]

Level sets of PDF:

If \( p = 0 \) then \( x_1 \).