- <u>Density (PDF) of multivariate normal distr:</u>

- Standard normal:

$$Z = (Z_1, \dots, Z_n) \sim N(0, I_n) \ :$$

$$\underset{\substack{\| \\ (z_1, \dots, z_n)}}{f_Z(z)} = f_Z(z_1, \dots, z_n) = f_{Z_1}(z_1) \cdots f_{Z_n}(z_n) \qquad \text{(indep.)}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-z_1^2/2} \cdots \frac{1}{\sqrt{2\pi}} e^{-z_2^2/2}$$

$$\boxed{f_Z(z) = \frac{1}{(2\pi)^{n/2}} e^{-|z|^2/2}}$$

where $|z|$ is the length of z.

<u>Rotation invariance.</u>

- <u>General normal:</u> ~~$X = \mu + AZ$,~~ $X \sim N(\mu, \Sigma)$

$$X = \mu + AZ.$$

- $E[X] = \mu$;
- $\Sigma = Cov(X) = E[(X-\mu)(X-\mu)^T] = E[(AZ)(AZ)^T]$

$$= E[AZZ^TA^T] = A \underbrace{E[ZZ^T]}_{\substack{\| \\ \Sigma(Z) = I}} A^T = AA^T.$$

- Transformation: If $x = (x_1, \dots, x_n)$, $z = (z_1, \dots, z_n)$, then

$$\left\{ \begin{array}{l} f_X(x) = f_Z(z) \cdot |J|^{-1} \quad \text{where} \quad J = \left[\frac{\partial x_i}{\partial z_j}\right] \\ \qquad\qquad \text{and} \quad x = \mu + az. \\ f_Z(z) = \frac{1}{(2\pi)^{n/2}} e^{-|z|^2/2}, \end{array} \right.$$

Here: $J = \det(A) = \sqrt{\det(AAT)} = \sqrt{\det \Sigma}$,

$\cancel{z = \quad |z|^2 = z^T z = (A^{-1}(x}$

$$z = A^{-1}(x-\mu), \quad \text{so}$$

$$|z|^2 = z^T z = \left(A^{-1}(x-\mu)\right)^T A^{-1}(x-\mu)$$

$$= (x-\mu)^T \underbrace{(A^{-1})^T A^{-1}}_{(AA^T)^{-1} = \Sigma^{-1}} (x-\mu)$$

$$= \cancel{(x-\mu)^T (AA^T)^{-1}}$$

Hence:

$$f_X(x) = \frac{1}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right] \cdot \frac{1}{\sqrt{\det(\Sigma)}}$$

$\Rightarrow$ we proved:

~~THM The PDF of X~~

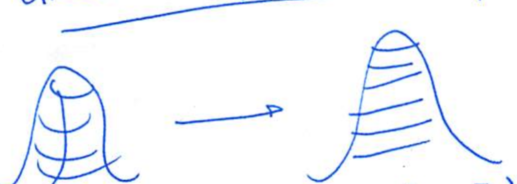
THM (Density of multivariate normal distr.)

$$\boxed{\text{The PDF of } X \sim N(\mu, \Sigma) \text{ is } \quad f(x) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp\left[-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right], \quad x \in \mathbb{R}^n.}$$

- Compare to univariate $X \sim N(\mu, \sigma^2)$:

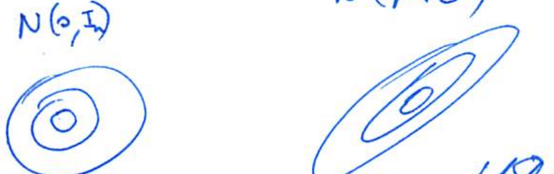
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-(x-\mu)^2/2\sigma^2\right)$$

- Linear transformation of standard normal PDF



$N(0, I_n)$ $\qquad$ $N(\mu, \Sigma)$

Level sets



Ex. $E[X_1] = E[X_2] = 0,$

$\mathrm{var}(X_1) = 1, \qquad \mathrm{var}(X_2) = 4,$

$\rho_{X_1,X_2} = 1/2 \quad = \dfrac{\mathrm{Cov}(X_1,X_2)}{\sqrt{\mathrm{var}\,X_1}\ \sqrt{\mathrm{var}\,X_2}} = \dfrac{\mathrm{Cov}(X_1,X_2)}{1\cdot 2}$

Compute PDF. $\mathrm{Cov}(X_1,X_2) = 1.$

$$\Sigma = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}. \qquad \Sigma^{-1} = \begin{bmatrix} 4/3 & -1/3 \\ -1/3 & 1/3 \end{bmatrix}.$$

$$\det(\Sigma) = 3.$$

$n=2$

$$f(x) = \frac{1}{2\pi\sqrt{3}} \exp\left(-\frac{1}{2} x^T \Sigma^{-1} x\right)$$

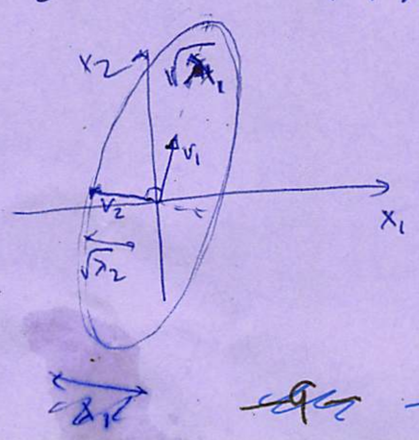
$$= \frac{1}{2\pi\sqrt{3}} \exp\left[-\frac{1}{2}\left(\frac{4}{3}x_1^2 + \frac{1}{3}x_2^2 - \frac{2}{3}x_1 x_2\right)\right]$$

$$= \frac{1}{2\pi\sqrt{3}} \exp\left(-\frac{4x_1^2 + x_2^2 - 2x_1x_2}{6}\right).$$

Eigenvalues / eigenvectors: $\lambda_1 \approx 4.3$, $v_1 \approx \begin{pmatrix} 0.3 \\ 1 \end{pmatrix}$

$\lambda_2 \approx 0.7$, $v_2 \approx \begin{pmatrix} -3.3 \\ 1 \end{pmatrix}$

Level sets of PDF:



If $\rho = 0$ then