8.4. The Law of Large Numbers.

"WITH PROBABILITY = 1" !

Let $X_1, X_2, \ldots$ be iid rv's, $\mu = E[X_i]$.

Then, with probability 1,

$$\frac{X_1 + \cdots + X_n}{n} \to \mu \quad \text{as } n \to \infty$$

$$\Rightarrow \quad P\left\{ \lim_{n \to \infty} \frac{X_1 + \cdots + X_n}{n} = \mu \right\} = 1.$$

"Almost surely" = "with probability 1".

Lemma (= Borel-Cantelli). Let $Z_n \geq 0$ be rv's.

If

$$\sum_{n=1}^{\infty} E[Z_n] < \infty$$

then, with prob. 1,

$$Z_n \to 0 \quad \text{almost surely}, \quad \text{as } n \to \infty$$

Assumption $\Rightarrow E\left[\sum_{n=1}^{\infty} Z_n\right] < \infty$.

$\Rightarrow$ with prob. 1, $\sum_{n=1}^{\infty} Z_n < \infty.$ (Otherwise $\sum_{n=1}^{\infty} Z_n = \infty$ with some prob $p > 0$)

$\Rightarrow E[Z_n] = 0.$
Proof of LLN. For simplicity assume
\[ E(X_i) = 0, \quad \text{Var}(X_i) = 1, \quad \text{Var}(X_i^4) \leq 10. \]

- \( S_n = X_1 + \ldots + X_n \).

WTS: with Prob. 1, \( \frac{S_n}{n} \to 0 \) ?

Try to use BC Lemma. But \( \frac{S_n}{n} \neq 0 \).

Square:
\[ E\left[ \left( \frac{S_n}{n} \right)^2 \right] = \frac{\text{Var}(S_n)}{n} = \frac{n}{n} = 1. \]

\[ \sum_{n=1}^{\infty} \frac{1}{n} = \infty! \quad \text{Not good.} \]

- Take 4th power.

\[ E\left( \frac{S_n^4}{n^4} \right) = ? \]

\[ S_n^4 = (X_1 + \ldots + X_n)(X_1 + \ldots + X_n)(X_1 + \ldots + X_n)(X_1 + \ldots + X_n) \]

Expand \( \Rightarrow \) terms \( X_i^4, \ X_i^2 X_j, \ X_i X_j^2, \ X_i X_j X_k, \ X_i X_j X_k X_l \) \( \ldots \)

\[ \forall i \neq j, \text{ there are } \binom{4}{2} = 6 \text{ terms } X_i^2 X_j^2. \]

2. \[ E\left( S_n^4 \right) = \frac{n \cdot E[X_i^4] + 6 \cdot \binom{n}{2} \cdot E[X_i^2] E[X_j^2]}{10 \text{ (assumed)}} \]

\[ \leq 10n + 3n(n-1) \leq 3n^2 + 10n \leq 13n^2. \]

2. \[ E\left( \frac{S_n^4}{n^4} \right) \leq \frac{13n^2}{n^6} \to 0 \text{ with prob. 1.} \]

Remark: "Moment method."
Implications of LLN.

1. Frequencies → probability.
   
   Flip a coin indefinitely.

   LLN: "frequency of heads → $\frac{1}{2}$ with probability 1."

   \[
   X_i = \begin{cases} 
   1 & \text{if head} \\
   0 & \text{if tail}
   \end{cases}
   \]

   Frequency of heads \(= \frac{X_1 + \ldots + X_n}{n} \xrightarrow{n \to \infty} \frac{1}{2} \text{ as } n \to \infty\)

   More generally:

   "Suppose an experiment results in a success with prob = p.

   Then if we repeat the experiment indefinitely, the frequency of successes converges to p with prob 1."

2. Infinite Monkey Theorem
   
   A monkey hitting keys at random on a computer keyboard will eventually type "Romeo and Juliet" almost surely with probability 1.

   Cast as a repeated experiment: every day, check if typed RJ (success)

   \[
   \text{Frequency with prob. freq.}
   \]

   \[
   p(\text{success}) = p > 0 \quad \Rightarrow \quad \text{with prob. 1,}
   \]

   \[
   \text{frequency of successes } \rightarrow p \quad \text{as } n \to \infty.
   \]

   \[
   \text{with prob. 1, there must be at least one success eventually (otherwise freq } = 0).
   \]

   Remark: 1. Will retype RJ \(\infty\) times!

   \[
   \Rightarrow p(\text{success}) = \frac{1}{130,000} \quad \Rightarrow \quad p = \frac{1}{20,000} \Rightarrow \text{E(time)} = 10,000.0\text{. (Compare to sec particles } = 10^{-9})
   \]