8.4. The Law of Large Numbers.

"WITH PROBABILITY = 1" !

THM Let $X_1, X_2, \dots$ be iid rv's, $\mu = E[X_i]$.
Then, with probability 1,

$$\frac{X_1 + \dots + X_n}{n} \to \mu \quad \text{as } n \to \infty$$

$$\Leftrightarrow \quad P\left\{ \lim_{n\to\infty} \frac{X_1 + \dots + X_n}{n} = \mu \right\} = 1.$$

"Almost surely" = "with probability 1".

Lemma ($\approx$ Borel-Cantelli). Let $Z_n \geq 0$ be rv's.

If $\sum_{n=1}^{\infty} E[Z_n] < \infty$

then, with prob. 1, $Z_n \to 0$ ~~almost surely~~ as $n \to \infty$

Assumption $\Rightarrow E\left[\sum_{n=1}^{\infty} Z_n\right] < \infty$.

$\Rightarrow$ with prob. 1, $\sum_{n=1}^{\infty} Z_n < \infty$. (Otherwise $\sum_n Z_n = \infty$ with some prob $p > 0$ $\Rightarrow E[\sum Z_n] = \infty$)

$\Downarrow$

$Z_n \to 0$.

Proof of LLN. For simplicity assume

$$E(X_i)=0, \quad Var(X_i)=1, \quad E(X_i^4)\le 10.$$

- $S_n := X_1+\dots+X_n$.

WTS: with prob. 1, $\frac{S_n}{n}\to 0$?

- Try to use BC lemma. But $\frac{S_n}{n}\not\to 0$. Square:

$$E\left[\left(\frac{S_n}{n}\right)^2\right] = Var\left(\frac{S_n}{n}\right) = \frac{Var(S_n)}{n^2} = \frac{n}{n^2} = \frac{1}{n}.$$

$$\sum_{n=1}^{\infty}\frac{1}{n} = \infty! \quad \text{Not good.}$$

- Take 4^{th} power.

$$E\left[\left(\frac{S_n}{n}\right)^4\right] = ?$$

$$S_n^4 = (X_1+\dots+X_n)(X_1+\dots+X_n)(X_1+\dots+X_n)(X_1+\dots+X_n)$$

$\quad i \quad i \quad j \quad j$

Expand $\Rightarrow$ terms X_i^4, $X_i^3X_j$, $X_i^2X_j^2$, $X_i^2X_jX_k$, $X_iX_jX_kX_\ell$

Expectation $=0$

$\forall\, i<j$, there are $\binom{4}{2}=6$ terms $X_i^2X_j^2$.

$$\Rightarrow E[S_n^4] = E\Big[\sum_{i=1}^n X_i^4 + 6\underbrace{\sum_{i<j}}_{\binom{n}{2}\text{ terms}} X_i^2X_j^2\Big]$$

$$= n\underbrace{E[X_i^4]}_{\le 10 \text{ (assume)}} + 6\binom{n}{2}\underbrace{E[X_i^2]}_{=1}\underbrace{E[X_j^2]}_{=1}$$

$$\le 10n + 3n(n-1) \le 3n^2+10n \le 13n^2.$$

$$\Rightarrow E\left[\left(\frac{S_n}{n}\right)^4\right] \le \frac{13n^2}{n^4} = \frac{13}{n^2}.$$

$$\sum_{n=1}^{\infty}\frac{13}{n^2}<\infty \xRightarrow{\text{BC lemma}} \left(\frac{S_n}{n}\right)^4 \to 0 \text{ with prob. 1.} \quad \square$$

Remark: "Moment method".

Implications of LLN

① Frequencies → probability.

Flip a coin indefinitely.

LLN: "frequency of heads $\to \frac{1}{2}$ with probability 1."

$X_i := \begin{cases} 1 & \text{if head} \\ 0 & \text{if tail.} \end{cases}$ Frequency of heads $= \dfrac{\#\text{heads in } n \text{ flips}}{n} = \dfrac{X_1 + \dots + X_n}{n} \to \dfrac{1}{2}$ as $n \to \infty$.

More generally:

"Suppose an experiment results in a success with prob $= p$.
~~Then f~~ If we repeat the experiment indefinitely,
the frequency of successes converges to p
with prob. 1"

② Infinite Monkey Thm

A monkey hitting keys at random on a computer keyboard will eventually type "Romeo and Juliet" ~~almost surely~~ with probability 1.

Cast as a repeated experiment: every day, check if typed RJ (success)

~~Frequenc~~ ~~with prob. 1, frequen~~

$P(\text{success}) = p > 0$. ⟹ with prob $= 1$,

⟹ frequency of successes in n days $\to p$ as $n \to \infty$.

⟹ with prob. 1, there must be at least one success eventually (otherwise freq $= 0$).

Remark: ① Will retype RJ ∞ times!

② $p = ?$ #letters in RJ $\approx 130{,}000$ Each letter correct with prob $= \frac{1}{30}$

$\Rightarrow p = \left(\frac{1}{30}\right)^{130{,}000} \approx 10^{-20{,}000}$ (Compare to #particles $\approx 10^{80}$).

$\Rightarrow E(\text{time}) \approx 10^{20{,}000}$.