Midterm Exam 1

Math 130B, Prof. Roman Vershynin
Winter 2018

Name: SOLUTIONS

Read the following information before starting the exam.

- No calculators or textbooks are allowed on this exam.
- Show all work, clearly and in order, if you want to get full credit. Points may be taken off if it is not clear how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. Points may be taken off for rambling and for incorrect or irrelevant statements.

1. (*10 points*) Let X and Y be discrete random variables with joint PMF (probability mass function) given by the table below.

		Y		
		1	2	3
X	1	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$
	2	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

To illustrate how to read the table, for example $P\{X = 2, Y = 1\} = 1/3$.

a. (*5 pts*) Find the marginal PMFs of X and Y.

$P_X(1) = \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{1}{3}$; $P_X(2) = \frac{1}{3} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$.

$P_Y(1) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$; $P_Y(2) = \frac{1}{12} + \frac{1}{6} = \frac{1}{4}$; $P_Y(3) = \frac{1}{12} + \frac{1}{6} = \frac{1}{4}$.

b. (*5 pts*) Are X and Y independent? Explain.

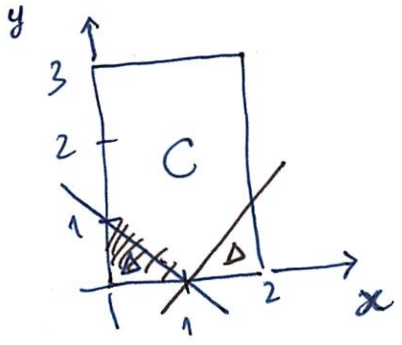
$p(x,y) \stackrel{?}{=} P_X(x) \cdot P_Y(y) \quad \forall x,y$

Yes : $p(1,1) = \frac{1}{6} = \frac{1}{3} \times \frac{1}{2} = P_X(1) \cdot P_Y(1)$

and similarly for all other entries.

2. (*10 points*) Let X be a random variable uniformly distributed in the interval $[0, 2]$, and Y be a random variable uniformly distributed in the interval $[0, 3]$. Suppose X and Y are independent. Compute

$$P\{Y > X - 1\}.$$



$$P\{Y > X - 1\} = \iint_C f(x,y)\,dx\,dy$$

$$f(x,y) = \frac{1}{2 \cdot 3} = \frac{1}{6} \text{ on } C.$$

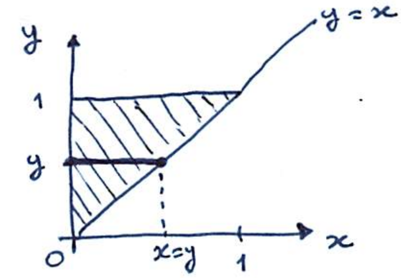
Thus $$P\{Y > X - 1\} = \frac{1}{6} \iint_C dx\,dy$$

$$= \frac{1}{6} \text{Area}(C) = \frac{1}{6}\left(2 \times 3 - \text{Area}(\Delta)\right)$$

$$= \frac{1}{6}\left(6 - \frac{1}{2}\right) = 1 - \frac{1}{12} = \frac{11}{12}.$$

3. (*10 points*) Let X and Y be random variables with joint PDF (probability density function) given by

$$f(x,y) = \begin{cases} 8xy & \text{if } 0 \le x \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$



a. (*5 pts*) Compute the marginal PDF of Y.

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^y 8xy\,dy = 8y \int_0^y x\,dx$$

$$= \begin{cases} 4y^3 & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

b. (*5 pts*) Compute $E[Y]$.

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy = \int_0^1 4y^4\,dy = \frac{4}{5}.$$

4. (*10 points*) Ford and GM pickup trucks are being tested for fuel efficiency, which is commonly measured in miles per gallon (mpg). The fuel efficiency of Ford trucks is found to follow a normal distribution with mean 20 mpg and standard deviation 3 mpg. The fuel efficiency of GM trucks is found to follow a normal distribution with mean 24 mpg and standard deviation 4 mpg. Compute the probability that a randomly chosen Ford truck outperforms a randomly chosen GM truck in terms of fuel efficiency.

$X :=$ Ford fuel efficiency; $Y :=$ GM fuel efficiency.

$X \sim N(20, 3^2)$, $Y \sim N(24, 4^2)$ are independent.

Thus $X - Y \sim N(20-24, 3^2+4^2) = N(-4, 5^2)$.

$$P\{X > Y\} = P\{X - Y > 0\} = P\left\{\frac{X-Y+4}{5} > \frac{4}{5}\right\}$$

$$= P\left\{Z > \frac{4}{5}\right\} \quad (\text{where } Z \sim N(0,1))$$

$$= 1 - \Phi(0.8) \approx 1 - 0.789 = 0.211$$

5. (*10 points*) A store is about to close for the day. There are still two (unrelated) customers in the store. We estimate that it takes 10 minutes for a customer to exit the store. What is the probability that it would take more than 20 minutes for the last customer to leave the store?

Let T_1, T_2 be the times the customers will remain in the store.

$T_1, T_2 \sim Exp(\lambda)$ independent.

$E[T_i] = \frac{1}{\lambda} = 10$, hence $\lambda = \frac{1}{10}$.

The last customer leaves at time $\max(T_1, T_2)$.

$$P\{\max(T_1,T_2) > 20\} = P\{T_1 > 20 \text{ or } T_2 > 20\}$$

$$= 1 - P\{T_1 \le 20, \text{ and } T_2 \le 20\}$$

$$= 1 - P\{T_1 \le 20\} \cdot P\{T_2 \le 10\} \quad \text{(By Independence)}$$

$$\cancel{=1-e} \; 1-(1-e^{-\lambda \cdot 20})^2 = 1-(1-e^{-2})^2.$$

(Total number of points in this exam = 50)

Discrete Distributions

Distrtibution	PMF	$E[X]$	$\mathrm{Var}(X)$	
Bernoulli(p)	$p(0) = 1 - p$, $p(1) = p$	p	$p(1-p)$	
Binom(n, p)	$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$	$(pe^t + 1 - p)^n$
Poisson(λ)	$p(k) = \frac{\lambda^k}{k!} e^{-\lambda}$	λ	λ	$\exp\{\lambda(e^t - 1)\}$
Geometric(p)	$p(k) = (1-p)^{k-1} p$	$1/p$	$\frac{1-p}{p^2}$	

Continuous Distributions

Distrtibution	PDF $f(x)$	CDF $F(x)$	$E[X]$	$\mathrm{Var}(X)$	MGF $M(t)$
Uniform(α, β)	$\frac{1}{\beta - \alpha}$, $\alpha \le x \le \beta$		$\frac{\alpha+\beta}{2}$	$\frac{(\beta-\alpha)^2}{12}$	
$N(\mu, \sigma^2)$	$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$	$\Phi(x)$: see table	μ	σ^2	$\exp(\mu t + \sigma^2 t^2/2)$
Exp(λ)	$\lambda e^{-\lambda x}$, $x \ge 0$	$1 - e^{-\lambda x}$, $x \ge 0$	$1/\lambda$	$1/\lambda^2$	$\frac{\lambda}{\lambda - t}$

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF X

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
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2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998