Midterm Exam 2

Math 130B, Prof. Roman Vershynin
Winter 2018

Name: SOLUTIONS

Read the following information before starting the exam.

- No calculators or textbooks are allowed on this exam.
- Show all work, clearly and in order, if you want to get full credit. Points may be taken off if it is not clear how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. Points may be taken off for rambling and for incorrect or irrelevant statements.

1. (*10 points*) Suppose the joint probability density function (PDF) of random variables X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{y}e^{-x/y}e^{-y}, & x,y \in (0,\infty) \\ 0, & \text{elsewhere.} \end{cases}$$

Find the conditional probability density function of X given $Y = 1$.

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{\infty} \frac{1}{y} e^{-x/y} e^{-y}\,dx$$

$$= \frac{e^{-y}}{y} \int_0^{\infty} e^{-x/y}\,dx = \begin{cases} e^{-y}, & y \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

$$f_{X|Y}(x|1) = \frac{f(x,1)}{f_Y(1)} = \frac{e^{-x}e^{-1}}{e^{-1}} = \boxed{\begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}}$$

2. (*10 points*) Suppose the joint probability density function (PDF) of random variables X and Y is given by

$$f(x, y) = \frac{c}{1 + |x^2 - y^2|}, \quad x, y \in \mathbb{R},$$

where c is a constant. Consider random variables

$$U = X + Y, \quad V = X - Y.$$

Compute the joint probability density function of U and V.

$$f_{U,V}(u, v) = f_{X,Y}(x, y) \, |J(x, y)|^{-1}$$

where $u = x + y$, $v = x - y$.

$$J(x, y) = \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2$$

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}.$$

Thus

$$f_{U,V}(u, v) = f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \cdot \frac{1}{2}$$

$$= \frac{c}{2\left(1 + \left|\frac{(u+v)^2}{4} - \frac{(u-v)^2}{4}\right|\right)}$$

$$= \frac{c}{2 + 2|uv|}, \quad u, v \in \mathbb{R}.$$

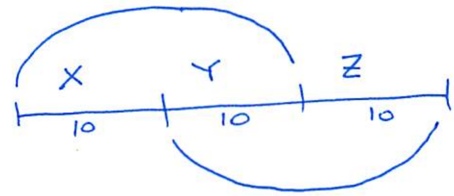
3. (*10 points*) Consider independent random variables $X \sim N(3, 2)$ and $Y \sim N(5, 3)$. (Thus, X is a normal random variable with mean 3 and variance 2, and likewise for Y.) Compute $\mathrm{Var}(XY)$.

$$\mathrm{Var}(XY) = E[X^2Y^2] - (E[XY])^2$$
$$= E[X^2]\,E[Y^2] - (E[X]\,E[Y])^2 \quad \text{By independence.}$$

$$E[X^2] = \mathrm{Var}(X) + (E(X))^2 = 2 + 3^2 = 11,$$
$$E[Y^2] = \mathrm{Var}(Y) + (E[Y])^2 = 3 + 5^2 = 28.$$

$$\mathrm{Var}(XY) = 11 \cdot 28 - (3 \cdot 5)^2 = 83$$

4. (*10 points*) A fair coin is flipped 30 times. Find the covariance between the numbers of heads among the first 20 flips and the last 20 flips.



Let $X = \#$ heads among first 10 flips
$Y = \#$ heads among middle 10 flips
$Z = \#$ heads among last 10 flips.

$$\mathrm{Cov}(X+Y,\ Y+Z) = \underset{0}{\underset{\|}{\mathrm{Cov}(X,Y)}} + \underset{0}{\underset{\|}{\mathrm{Cov}(X,Z)}} + \mathrm{Cov}(Y,Y) + \underset{0}{\underset{\|}{\mathrm{Cov}(Y,Z)}}$$

(by independence)

$$= \mathrm{Var}(Y) = np(1-p)$$

(since $Y \sim \mathrm{Binom}(10, \frac{1}{2})$)

$$= 10 \cdot \frac{1}{2} \cdot \frac{1}{2} = 2.5$$

5. (*10 points*) 10 tennis players participate in a round robin tournament. This means that every player plays everybody else exactly once. (Hence each player plays 9 matches.) Assume that the outcomes of the matches are random: in each match each of the two players wins with probability 1/2 independently of the outcomes of the other matches. We say that three players A, B, C form a *cycle* if A beats B, B beats C, and C beats A. Find the expected number of cycles.

$\underbrace{\text{number of cycles}}_{= X}$

or vice versa: A is beaten by B, B is beaten by C, and C is beaten by A.

$$X = \sum_T X_T \quad \text{where} \quad X_T = \begin{cases} 1, & \text{if the triple } T=(ABC) \text{ is a cycle} \\ 0, & \text{otherwise.} \end{cases}$$

Here the sum is over all $\binom{10}{3}$ triples $T=(ABC)$ of players.

$$E[X_T] = P\{X_T = 1\} = P\{\text{a given triple } T=(ABC) \text{ is a cycle}\}$$

$$= P\{ A \xrightarrow{\text{beats}} B \to C \to A \ \text{ or } \ A \leftarrow B \leftarrow C \leftarrow A \}$$

$$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 = \frac{1}{4}.$$

Thus $$E[X] = \sum_T E[X_T] = \binom{10}{3} \cdot \frac{1}{4} = \frac{10\cdot 9\cdot 8}{2\cdot 3}\cdot\frac{1}{4} = 30.$$

(Total number of points in this exam = 50)