Random Vectors.

R.Vs $X_1 \dots X_n$

Useful to view

$$X = (X_1, \dots, X_n) \in \mathbb{R}^n$$

as a (random) vector in $\mathbb{R}^n$.

- **Notion of mean**: $E[X] := \left(E[X_1], \dots, E[X_n] \right)$.
- **Notion of variance**? Replaced with: ~~The~~

Def Covariance matrix of a r.v. $X = (X_1, \dots, X_n)$ is the $n \times n$ matrix

$$\boxed{\Sigma = \Sigma(X) = \left[\mathrm{Cov}(X_i, X_j) \right]_{i,j=1}^{n}}$$

~~Eg Example~~

e.g. for $n=2$: $\Sigma(X) = \begin{bmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1, X_2) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Var}(X_2) \end{bmatrix}$

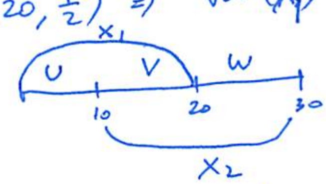
- Symmetric, PSD. Diagonal iff X_i's are uncorrelated.

Ex: Flip a coin 30 times. $X_1 = \#$ heads in first 20 flips

$X_2 = \#$ heads in last 20 flips.

$X = (X_1, X_2)$.

$X_1, X_2 \sim \mathrm{Binom}(20, \tfrac{1}{2}) \Rightarrow \mathrm{Var}(X_1) = \mathrm{Var}(X_2) = 20 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = 5.$



$X_1 = U + V$

$X_2 = V + W$.

$$\mathrm{Cov}(X_1, X_2) = \mathrm{Cov}(U+V, V+W) = \underbrace{\mathrm{Cov}(U,V)}_{=0} + \underbrace{\mathrm{Cov}(V,V)}_{=\mathrm{Var}(V)} + \underbrace{\mathrm{Cov}(U,W)}_{=0} + \underbrace{\mathrm{Cov}(V,W)}_{=0}$$

$$= \mathrm{Var}(V), \quad V \sim \mathrm{Binom}(10, \tfrac{1}{2})$$

$$= 10 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = 2.5.$$

$$\Sigma = \begin{bmatrix} 5 & 2.5 \\ 2.5 & 5 \end{bmatrix}.$$

Prop $\boxed{\Sigma(X) = E[(X-\mu)(X-\mu)^T] \quad \text{where } \mu = E[X]}$

Recall: outer product of $x, y \in \mathbb{R}^n$ is the $n \times n$ matrix $xy^T = \begin{bmatrix} x_1y_1 & \cdots & x_1y_n \\ & & \\ x_ny_1 & \cdots & x_ny_n \end{bmatrix} = [x_iy_j]_{i,j=1}^n$.

The (i,j) entry of $(X-\mu)(X-\mu)^T$ is

$(X_i - \mu_i)(X_j - \mu_j)$.

Take expectation.. $E[(X_i - \mu_i)(X_j - \mu_j)] = \mathrm{Cov}(X_i, X_j)$

$=$ (i,j) entry of LHS

- Compare to univariate: $\mathrm{Var}(X) = E[(X-\mu)^2]$

- MGF in vector notation:

~~M(t)~~ If $X=(X_1,\dots X_n)$, $t=(t_1\dots t_n)$

then ~~t.~~ $t_1X_1+\dots+t_nX_n = t\cdot x = t^Tx = \langle t, x\rangle$

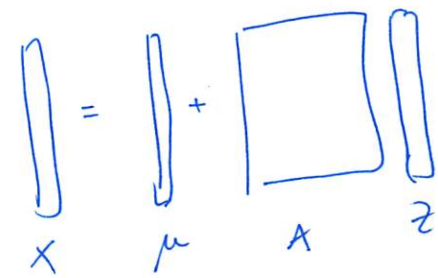
$$\boxed{M(t) = E\left[e^{t\cdot X}\right], \quad t\in\mathbb{R}^n}$$

Compare with $M(t)=E\left[e^{tx}\right]$ for r.variables.

- Multivariate normal (in vector notation):

$Z=(Z_1,\dots,Z_n)\sim \boxed{N(0, I_n)}$

$X=\mu+AZ$ where $\mu\in\mathbb{R}^n$, $A: n\times n$



~~$\partial(M(t)$~~ $\Rightarrow$ Rewrite formula for MGF p.3:

$$\boxed{M(t) = \exp\left(t\cdot\mu + \tfrac{1}{2}\, t^T \cancel{A}\,\Sigma\, t\right)}$$

where $\Sigma=\Sigma(X)$ is the covariance matrix of X.

- Compare with MGF of $N(\mu,\sigma^2)$: $M(t)=\exp\left(t\mu+\frac{1}{2}t^2\sigma^2\right)$

Rewrite THM.:

THM The multivariate distribution is completely determined by its mean μ and covariance matrix Σ.

THUS, denoted $\boxed{X\sim N(\mu,\Sigma)}$

- Standard normal: ~~$Z\sim N(\mu, I$~~ $Z\sim N(0, I_n)$ identity.