Final Exam

Math 451, Prof. Roman Vershynin
Fall 2011

Name: SOLUTIONS

Read the following information before starting the exam:

- No laptops or any communication devices are allowed on the exam.
- Show all work, clearly and in order, if you want to get full credit. Points will be taken off if it is not clear how you arrived at your answer (even if your final answer is correct).
- Please keep your written answers brief; be clear and to the point. Points may be taken off for rambling and for incorrect or irrelevant statements.

1. (*23 points*) Compute the following limits or prove that they do not exist. Justify.

a. (*8 pts*) $\lim_n \dfrac{2^{n+1}+3^{n+1}}{2^n+3^n} = \lim_n \dfrac{2\cdot 2^n + 3\cdot 3^n}{2^n+3^n} = \lim_n \dfrac{2\cdot(2/3)^n+3}{(2/3)^n+1} \; (=)$

Since $(2/3)^n \to 0$ $(2/3 < 1)$, we have

$(=) \lim_n \dfrac{2\cdot 0+3}{0+1} = \boxed{3}$.

b. (*7 pts*) $\lim_{x\to 0} \dfrac{x^2\sin(1/x)}{\sin x} = \lim_{x\to 0}\left(\dfrac{x}{\sin x}\cdot x\sin\dfrac{1}{x}\right) \; (=)$

1) $\dfrac{x}{\sin x} \to 1$ as $x\to 0$

2) $x\sin\dfrac{1}{x} \to 0$ as $x\to 0$ since $|\sin\frac{1}{x}|\le 1$; the claim then follows by Squeeze Thm

Hence $(=)\; 1\cdot 0 = \boxed{0}$.

(Problem 1 continued)

c. (*8 pts*) $\lim_{x \to 0_+} x^x$

$x^x = e^{x \ln x}$, so first we compute $\lim_{x\to 0_+} x \ln x$.

Use L'Hopital's Rule for

$$x \ln x = \frac{\ln x}{1/x}, \quad \text{where} \quad \ln x \to -\infty, \ 1/x \to +\infty \quad \text{as } x \to 0_+$$

Since $\frac{(\ln x)'}{(1/x)'} = \frac{1/x}{-1/x^2} = -x \to 0$ as $x \to 0_+$.

By L'Hopital's Rule, $x \ln x \to 0$ as $x \to 0$.

By continuity of $f(y) = e^y$, it follows that

$$e^{x \ln x} \to e^0 = \boxed{1}$$

Answer: $\boxed{1}$.

2. (*10 points*) Find all values of $p > 0$ for which the series

$$\sum_{n=1}^{\infty} \sin \frac{1}{n^p}$$

converges.

Use Limit Comparison Theorem:

Since $\frac{\sin x}{x} \to 1$ as $x \to 0$

and $1/n^p \to 0$ as $n \to \infty$, we have

$$\frac{\sin 1/n^p}{1/n^p} \to 1 \quad \text{as } n \to \infty$$

Now, since $\sum_n \frac{1}{n^p}$ converges, ~~we conclude if and only~~ for $p > 1$ and diverges for $p \le 1$,

we conclude that

$$\boxed{\sum \sin \frac{1}{n^p} \text{ converges for } p > 1 \text{ and diverges for } p \le 1.}$$

3. (*12 points*) Consider a sequence (s_n) and a real number M. Prove that

$$\limsup s_n \leq M$$

if and only if there exists N such that

$$s_n \leq M \quad \text{for all } n > N. \qquad (*)$$

($\Rightarrow$: Necessity). If $\limsup s_n = \lim_{N \to \infty} \underbrace{\sup\{s_n : n > N\}}_{= a_N} \leq M$

then there exists N such that ~~sup~~

~~$a_n \leq M$ for all $n > N$ (by Ex. 8.9)~~

$a_N \leq M$

(otherwise $a_N > M$ for all N $\Rightarrow$ $\lim a_N \geq M$). QED.

(Alternative proof: by contradiction. Assume the conclusion fails; then $\exists$ subsequence $s_{n_k} > M$.
~~$\Rightarrow \limsup s_{n_k}$~~ $\Rightarrow a_N > M$ for all N $\Rightarrow$ $\limsup s_n = \lim a_N \geq M$.).

($\Leftarrow$) Sufficiency). If $\limsup s_n \leq M$ fails, ie $\limsup s_n > M$

then $\exists$ subsequence $s_{n_k} \to \limsup s_n > M$ (~~Thm~~ Cor. 11.4)

$\Rightarrow \exists\, K : s_{n_k} \geq M$ for all $k > K$.

This contradicts $(*)$

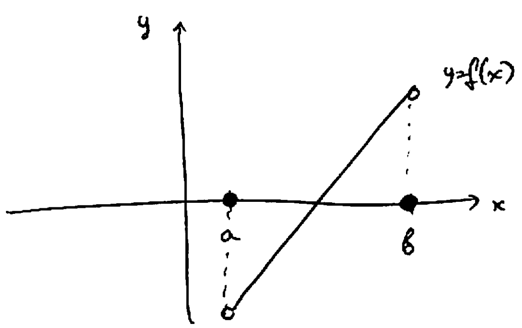
(Alternative proof: If $s_n \leq M$ for all $n > N$ $\Rightarrow$ $a_{N_1} \leq M$ for all $N_1 > N$
$\Rightarrow$ ~~Ba~~ $\limsup s_n = \lim_{N_1} a_{N_1} \leq M$.
QED)

4. (*12 points*) Let f be a function on $[a, b]$ and $a < c < b$. Let f_1 be the restriction of f onto $[a, c]$ (i.e. f_1 is the function f defined on $[a, c]$), and let f_2 be the restriction of f onto $[c, b]$.

Are the following statements true or false? Proofs are **not** necessary.

- **a.** (*4 pts*) If f is continuous then f_1 and f_2 are continuous. T
- **b.** (*4 pts*) If f_1 and f_2 are continuous then f is continuous. F
- **c.** (*2 pts*) If f is integrable then f_1 and f_2 are integrable. T
- **d.** (*2 pts*) If f_1 and f_2 are integrable then f is integrable. T

5. (*9 points*) Give an example of a function f on $[a, b]$, which is continuous on (a, b) but such that f attains neither its supremum nor infimum on $[a, b]$.



Why does this example not contradict the validity of Extreme Value Theorem?

Because the function is not continuous on [a, b].

(it is discontinuous at a, b)

6. (*12 points*) Let f be a differentiable function on $[0, \infty)$, which satisfies

$$\lim_{x \to +\infty} f'(x) = 0.$$

Prove that

$$L := \lim_{x \to +\infty} \Big(f(x+1) - f(x) \Big) = 0.$$

By Mean Value Thm, $\forall x \quad \exists \; \xi = \xi(x) \in [x, x+1]$:

$$f(x+1) - f(x) = f'(\xi(x)).$$

~~Since~~ Since $\xi(x) \to +\infty$ as $x \to +\infty$,

$f'(\xi(x)) \to 0$ by the assumption.

It follows that

$$L = \lim_{x \to +\infty} f'(\xi(x)) = 0.$$

7. (*12 points*) Let f be an integrable function on $[a, b]$. Prove that there exists $x \in [a, b]$ such that

$$\int_a^x f(t)\, dt = \int_x^b f(t)\, dt.$$

Define $F(x) = \int_a^x f(t)dt$, $G(x) = \int_x^b f(t)dt$.

F, G continuous by F.T.C.

$$\left.\begin{array}{l} F(a) = 0, \quad F(b) = \int_a^b f(t)dt \\ G(a) = \int_a^b f(t)dt, \quad G(b) = 0. \end{array}\right\} \Rightarrow (F-G)(a) = -(F-G)(b).$$

~~By~~ Opposite signs

$\Rightarrow$ ~~[illegible]~~ by Intermediate Value Thm, $\exists\ x \in [a, b]$ s.t.

$(F-G)(x) = 0 \Rightarrow F(x) = G(x)$. QED.

8. (*10 points*) Prove the following inequality:

$$\sin x \geq x - \frac{x^3}{6} \qquad \text{for } 0 \leq x \leq \pi.$$

Taylor's Thm $\Rightarrow$

$$\sin x = x - \frac{x^3}{3!} + \underbrace{\frac{\sin^{(4)}(\xi)}{4!}x^4}_{=R} \qquad \text{~~where~~ for some } 0 \leq \xi \leq x \leq \pi$$

Since $\sin^{(4)}(\xi) = \sin \xi \geq 0$ for $0 \leq \xi \leq \pi$,

$\Rightarrow$ $R > 0$. QED.

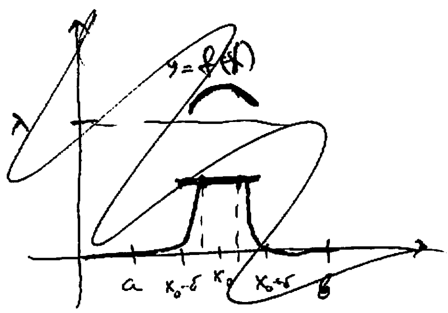
9. (*10 points*) **[Bonus problem, no partial credit]** Let f be a continuous function on $[a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = 0$$

for every continuous function g on $[a, b]$ that satisfies $g(a) = g(b) = 0$. Prove that

$$f(x) = 0 \quad \text{for all } x \in [0, 1].$$

(Hint: argue by contradiction; construct a suitable g.)



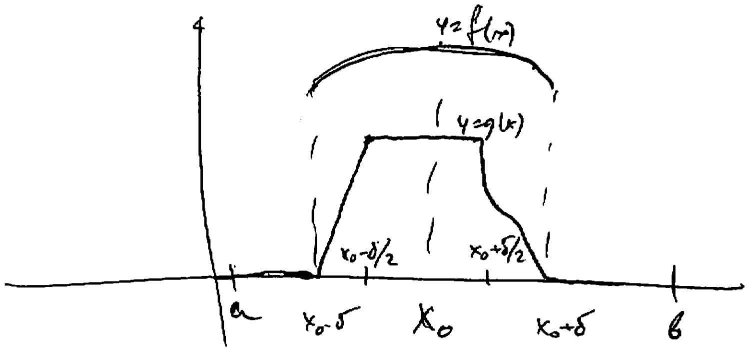
~~Pf~~ By contradiction.

WLOG Assume $\exists\, x_0 \in (0,1)$: $f(x_0) = \lambda > 0$.

$\Rightarrow \exists\, \delta$: $f(x) > \lambda/2$ for all $x_0-\delta < x < x_0+\delta$

Let $g(x)$: continuous,

$$\begin{cases} g(x)=1 & \text{for } x_0-\frac{\delta}{2} < x < x_0+\delta/2 \\ g(x)>0 & \text{for } x_0-\delta < x < x_0+\delta \\ g(x)=0 & \text{elsewhere.} \end{cases}$$



$$\Rightarrow \int_a^b f(x)g(x) =$$

$$= \underbrace{\int_a^{x_0-\delta}}_{0} + \underbrace{\int_{x_0-\delta}^{x_0+\delta}}_{>0} + \underbrace{\int_{x_0+\delta}^{b}}_{0} > 0, \text{ ↯}$$

$(f>0, g>0)$

QED