MAT-127A-2 FINAL EXAM March 23, 2004

2 hours, 7 problems, closed books, closed notes, no calculators

1. `[9 points]` Give $\liminf(s_n)$ and $\limsup(s_n)$ for the following sequences (s_n) (no proofs are needed).

(a) `[3 points]` $(-1)^n(1-\frac{1}{n})$

(b) `[3 points]` $\sin(\frac{\pi n}{3})$

(c) `[3 points]` $(-1)^n n - n$

2. `[12 points]` Assume that $\lim_{x\to 1^+} f(x) = 0$ and that $f(x) > 0$ for all $x \in \mathbb{R}$. Find $\lim_{x\to 1^+} \frac{1}{f(x)}$ and prove your assertion.

3. `[18 points]` Let

$$f(x) = \begin{cases} -x & \text{for } x < 1 \\ x & \text{for } x \geq 1. \end{cases}$$

Prove or disprove the following statements:

(a) f is uniformly continuous on $(-\infty, 1)$

(b) f is uniformly continuous on $(1, \infty)$

(c) f is uniformly continuous on $(1, \infty) \cup (-\infty, 1)$.

4. `[9 points]` Prove that $x^3 = \cos x$ for some x in $(0, \frac{\pi}{2})$.

5. `[12 points]` Prove that the function $f(x) = \ln x - \frac{x-1}{e-1}$ attains its maximum on $(1, e)$.

6. `[20 points]` Let f be a function such that

$$|f(x)| \leq x^2 \quad \text{for all } x \in [-1, 1].$$

Prove that f is differentiable at 0 and $f'(0) = 0$.

7. `[20 points]` Let (s_n) and (t_n) be two sequences such that (s_n) is unbounded and (t_n) is convergent (to a real number).

(a) Prove that the sequence $(s_n + t_n)$ is unbounded.

(b) Will the result in (a) remain always true if (t_n) diverges to $+\infty$ (instead of being convergent)? *(Prove this or find an example showing that this fails. Just a correct guess will not earn eny credit)*