Midterm Exam 1

Math 451, Prof. Roman Vershynin
Fall 2011

Name: ______________________________

Read the following information before starting the exam:

- No laptops or any communication devices are allowed on the exam.
- Show all work, clearly and in order, if you want to get full credit. Points may be taken off if it is not clear how you arrived at your answer (even if your final answer is correct).
- Please keep your written answers brief; be clear and to the point. Points may be taken off for rambling and for incorrect or irrelevant statements.

Problem	Points
1	
2	
3	
4	
5	
6 (bonus)	
Total	

1. (*10 points*) For each of the following sequences, compute $\inf\{s_n\}$, $\sup\{s_n\}$, $\liminf s_n$ and $\limsup s_n$. No justification is necessary; you may just write down the answers.

a. (*5 pts*) $\quad s_n = \dfrac{(-1)^n}{n} + \dfrac{1+(-1)^n}{2}$

b. (*5 pts*) $\quad s_n = -n\big[2+(-1)^n\big]$

2. (*20 points*) Let (s_n) be a convergent sequence and let $\lim s_n = s$. Show that $(|s_n|)$ is a convergent sequence and $\lim |s_n| = |s|$. Use the definition of limit in your argument.

3. (*20 points*) Consider nonempty bounded subsets A and B of positive real numbers, and define

$$A \cdot B = \{a \cdot b : \; a \in A, \; b \in B\}.$$

(That is, a number z is in $A \cdot B$ if $z = ab$ for some $a \in A$ and $b \in B$.) Show that

$$\sup(A \cdot B) = \sup A \cdot \sup B.$$

4. (*30 points*) Consider the sequence (a_n) defined recursively as

$$a_1 = 1, \qquad a_n = \sqrt{3a_{n-1} - 2} \quad \text{for } n = 2, 3, \ldots.$$

a. (*10 pts*) Show by induction that

$$1 \leq a_n \leq 2 \quad \text{for all } n = 1, 2, \ldots$$

b. (*10 pts*) Show by induction that the sequence (a_n) is non-decreasing.

c. (*10 pts*) Deduce that the sequence (a_n) converges and find its limit.

5. (*20 points*) Compute the following limits. Justify all steps. If you apply a theorem on the limit of a sum, product, or ratio, you may put "by a limit theorem" without specifying its number. Other theorems need to be specified (the theorem's number in the book or the page number in class notes is OK).

a. (*10 pts*) $\lim \dfrac{7n - \sin(\pi n^2) + 1}{3n + \cos(\pi n^2)}$

b. (*10 pts*) $\lim \left(\sqrt{n + \sqrt{n}} - \sqrt{n}\right)$

6. (*10 points*) [**Bonus problem, no partial credit**] Let a and b be positive real numbers. Compute

$$\lim \sqrt[n]{a^n + b^n}.$$