Midterm Exam 1. Math 451, Fall 2015, Prof. Vershynin

1. *(10 points)* Let S be a subset of $\mathbb{R}$. Suppose S is bounded above, and some upper bound u is an element of S. Prove that $\sup S = u$.

Solution. We have to show that:
(i) $\forall\, s \in S$, one has $s \leq u$;
(ii) $\forall\, u' < u\ \exists\, s' \in S$ such that $s' > u'$.
Statement (i) is nothing but the assumption that u is an upper bound of S. As for statement (ii), we can choose $s' := u$ to see that it is true. Q.E.D.

2. *(10 points)* For each of the following statements, decide if it is true or false. Prove or give a counterexample.

(a) *(5 points)* There exists a sequence of irrational numbers which converges to a rational number.

(b) *(5 points)* There exists a sequence that has a bounded subsequence but has no convergent subsequences.

Solution. (a) This is true. Indeed, let $a > 0$ be an irrational number, and let $x_n := a/n$ for $n \in \mathbb{N}$. Then (x_n) is a sequence of irrational numbers that converges to 0. Q.E.D.

(b) This is false. Indeed, assume that (x_n) contains a bounded subsequence. Then, by Bolzano-Weierstrass Theorem, this subsequence must contain a convergent subsequence. Q.E.D.

3. *(10 points)* Compute the limit

$$\lim \left(\sqrt{4n^2 + n} - 2n \right).$$

Solution. Multiplying and dividing by the conjugate, we obtain

$$\begin{aligned}\sqrt{4n^2+n}-2n &= \frac{(\sqrt{4n^2+n}-2n)(\sqrt{4n^2+n}+2n)}{\sqrt{4n^2+n}+2n} \\ &= \frac{n}{\sqrt{4n^2+n}+2n} = \frac{1}{\sqrt{4+1/n}+2} =: x_n.\end{aligned}$$

Since $\lim(1/n) = 0$, using the limit theorems we obtain that

$$\lim x_n = \frac{1}{\sqrt{4+0}+2} = \frac{1}{4}.$$

4. *(10 points)* Let (x_n) be a sequence that converges to a non-zero limit. Prove that all except finitely many terms x_n are non-zero.

Solution. Assume the contrary, namely that there are infinitely many zero terms x_n. In other words, there exists a subsequence (x_{n_k}) such that $x_{n_k} = 0$ for all $k \in \mathbb{N}$. Then $\lim x_{n_k} = 0$. On the other hand, since $\lim x_n = x \neq 0$, the limit of any subsequence (x_{n_k}) must be $x \neq 0$. This is a contradiction. Q.E.D.

5. *(10 points)* Let (x_n) be an increasing sequence and (y_n) be a decreasing sequence. Assume that $x_n \le y_n$ for all n. Prove that both sequences converge.

Solution. The assumptions imply that

$$x_1 \le x_n \le y_n \le y_1 \quad \text{and} \quad y_1 \ge y_n \ge x_n \ge x_1 \quad \forall\, n \in \mathbb{N}.$$

Thus both sequences (x_n) and (y_n) are monotone and bounded. By Monotone Convergence Theorem, they converge. Q.E.D.

6. *(10 points)* Prove that

$$\lim \frac{n!}{n^n} = 0.$$

Solution. Note that

$$\frac{n!}{n^n} = \frac{1 \cdot 2 \cdots n}{n \cdot n \cdots n} = \frac{1}{n} \cdot \left(\frac{2 \cdots n}{n \cdots n}\right) \leq \frac{1}{n} \quad \forall\, n \in \mathbb{N},\ n \geq 2.$$

Since $\lim(1/n) = 0$, the Squeeze Theorem implies that $\lim(n!/n^n) = 0$. Q.E.D.