LECTURE 10

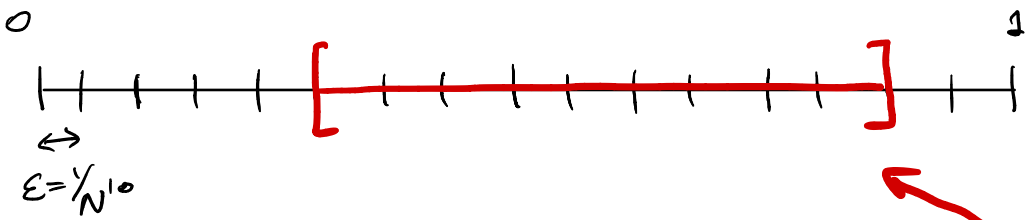
Proof of the Discrepancy Theorem

For simplicity, in dimension 1 **(2D → HW)**

So, instead of throwing points in the square $[0,1]^2$, we throw them onto the interval $[0,1]$.

- "ε-net method"

Choose an ε-net of $[0,1]$ with $\varepsilon = 1/N^{10}$: $\{0, \frac{1}{N^{10}}, \frac{2}{N^{10}}, \dots, 1\}$



"net interval" = interval whose endpoints $\in$ net.

There are $N^{10} \cdot N^{10} = N^{20}$ net intervals. (not ∞ anymore! ☺)

① Concentration. Fix any net interval I, length λ_I

\# points in it $N_I \sim$ Binomial with mean $\lambda_I N$

$\Rightarrow$ By Chernoff's inequality (small deviations – lec. 8):

$$\mathbb{P}\{ |N_I - \lambda_I N| \geq \underbrace{\delta \lambda_I N}_{\|} \} \leq 2\exp\left(-\frac{\delta^2 \lambda_I N}{6}\right) \quad \forall \delta \in [0,1].$$

Choose δ so that $\frac{C}{2}\sqrt{\lambda_I N \log N} \Rightarrow \delta = \frac{C\sqrt{\lambda_I N \log N}}{2\lambda_I N} \in [0,1]$ by assumption that LHS ≥ 0

$\delta^2 \lambda_I N = \frac{C^2}{4} \log N \Rightarrow$

$$\mathbb{P}\{ \underbrace{|N_I - \lambda_I N| \geq \frac{C}{2}\sqrt{\lambda_I N \log N}}_{} \} \leq 2\exp\left(-\frac{C^2 \log N}{4 \cdot 6}\right) \leq \frac{1}{N^{100}}$$

if we choose constant C large enough (e.g. $C = 1000$)

$\Rightarrow$ B_I, the event that interval I is bad, is unlikely.

② Union Bound:

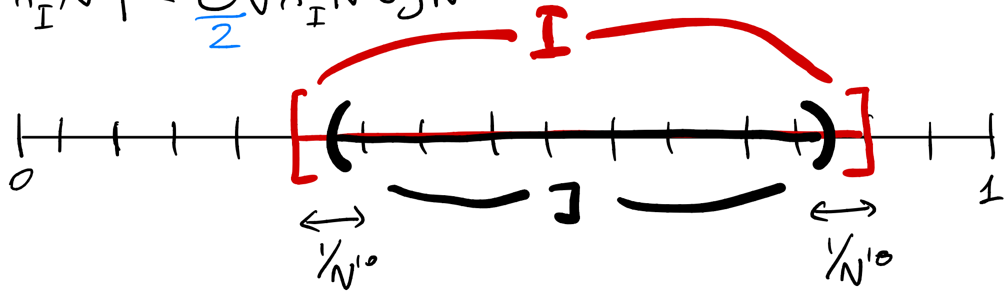
$$P\{\exists \text{ bad net-interval } I\} = P\Big(\bigcup_I B_I\Big) \le \underbrace{\sum_I}_{N^{20}\text{ terms}} \underbrace{P(B_I)}_{\le \frac{1}{N^{100}}}$$

$$\le \frac{N^{20}}{N^{100}} \le 0.01.$$

$\Rightarrow$ with probability ≥ 0.99, all net-intervals I are good,

i.e. $$|N_I - \lambda_I N| < \frac{C}{2}\sqrt{\lambda_I N \log N} \quad \forall \text{ net interval } I \quad (*)$$

③ Approximation:



- $\forall$ interval J lies in a net interval I with length
$$\lambda_I \le \lambda_J + \frac{2}{N^{10}}.$$

$$\Rightarrow \quad N_J \le N_I \le \lambda_I N + \frac{C}{2}\sqrt{\lambda_I N \log N} \qquad (\text{since } I \text{ is good}: (*))$$

$$\le \Big(\lambda_J + \frac{2}{N^{10}}\Big)N + \frac{C}{2}\sqrt{\Big(\lambda_J + \frac{2}{N^{10}}\Big) N \log N} \qquad \sqrt{a+b} \le \sqrt{a} + \sqrt{b}$$

$$\le \lambda_J N + \frac{C}{2}\sqrt{\lambda_J N \log N} + \underbrace{\frac{2}{N^9} + \frac{C}{2}\sqrt{\frac{2\log N}{N^9}}}_{= E} \quad \checkmark$$

- Finally, we can bound error E as follows:

$$E \le 1 \le \frac{C}{2}\sqrt{\lambda_J N \log N},$$

where the first inequality follows from the assumption that $N \ge C_1$ (with the absolute const. C_1 large enough), and the second inequality follows from the assumption that $LHS \ge 0$, i.e.
$\lambda_J N \ge C\sqrt{\lambda_J N \log N} \Rightarrow \sqrt{\lambda_J N} \ge C\sqrt{\log N} \Rightarrow \frac{C}{2}\sqrt{\lambda_J N \log N} \ge \frac{C^2}{2}\log N \ge 1$ if C is chosen large enough. QED