LECTURE 16

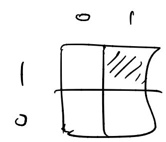
COMBINATORIAL OPTIMIZATION

- Problem: Given $(a_i)_{i=1}^n$, find $\max\limits_{x_i \in \{\pm 1\}} \sum_{i=1}^n a_i x_i$

Although exhaustive search is hard (2^n configurations of (x_i)),
$\exists$ direct solution: $x_i = \mathrm{sign}(a_i)$

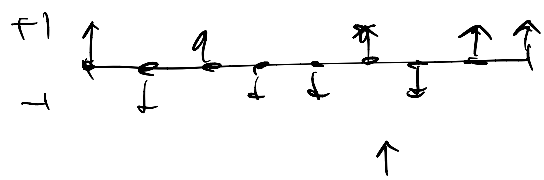
- Problem: Given $(a_{ij})_{i,j=1}^n$, find $\boxed{\max\limits_{x_i, y_i \in \{\pm 1\}} \sum_{i,j=1}^n a_{ij} x_i y_j}$ (1)

NP-hard. Integer quadratic program.



- Related NP-hard problems:

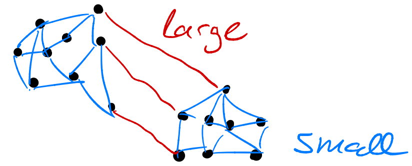
(2) $\boxed{\max\limits_{x_i \in \{\pm 1\}} \sum_{ij} a_{ij} x_i x_j}$



Ising model of magnetism: x_i = state of atom (spin ± 1)
a_{ij} = interaction between atoms i, j
($a_{ij} > 0 \Leftrightarrow$ ferromagnetic)
$H(x) = \sum_{ij} a_{ij} x_i x_j$ = Hamiltonian of the system (free energy)

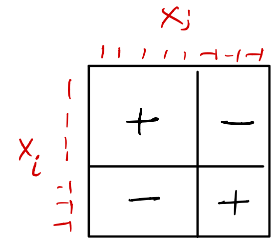
Maximal energy of the system

③ <u>Clustering</u> of points $v_1, \dots, v_n \in \mathbb{R}^d$



$a_{ij} := \|x_i - x_j\|_2$

$$\boxed{\min_{x_i \in \{\pm 1\}} \sum_{ij=1}^{n} a_{ij}\, x_i x_j} = \begin{array}{l} \Sigma \text{distances in cluster 1} \\ + \Sigma \text{distances in cluster 2} \\ - 2\Sigma \text{distances between the clusters} \end{array}$$



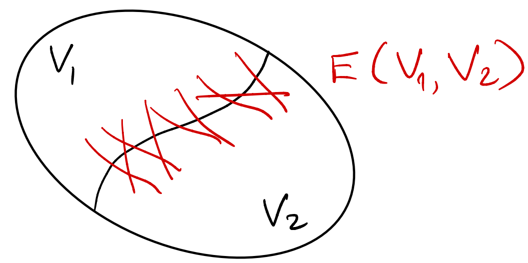
$\downarrow$

tries to make in-cluster distances small, across-cluster distances large.

- Equivalent to $\max \sum (-a_{ij})\, x_i x_j$ = ②
- Works in $\forall$ metric space.

[Frieze-Jerrum '1997]

④ <u>Max-cut</u> of graph $G=(V,E)$



$$\text{MaxCut}(G) = \max_{V = V_1 \sqcup V_2} |E(V_1,V_2)|.$$

- Analytically: WLOG $V = \{1,\dots,n\}$; $A = [a_{ij}]$ is the <u>adjacency matrix</u>

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{if not} \end{cases} \qquad x_i = \begin{cases} 1 & \text{if } i \in V_1 \\ -1 & \text{if } i \in V_2 \end{cases} \qquad \Rightarrow 1 - x_i x_j = \begin{cases} 2 & \text{if } (i,j) \in E(V_1,V_2) \\ 0 & \text{if not} \end{cases}$$

$E(V_1,V_2)$ = # edges between vertices of opposite signs

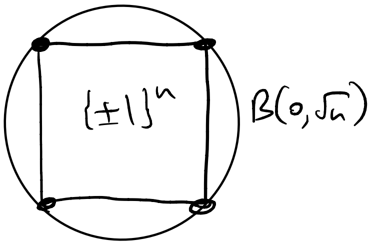
$$= \frac{1}{4} \sum_{i,j=1}^{n} a_{ij}\,(1 - x_i x_j)$$

(extra factor 2 to avoid double counting of (ij), (ji))

$$\boxed{\text{MaxCut}(G) = \max_{x_i \in \{\pm 1\}} \sum_{i,j=1}^{n} a_{ij}\,(1 - x_i x_j)}$$

$$= \sum a_{ij} + \max_{x_i \in \{\pm 1\}} \sum (-a_{ij})\, x_i x_j \quad \Rightarrow \text{equivalent to ②.}$$

SPECTRAL RELAXATION



Sp.R of ②: $\quad x \in \{-1,1\}^n \longrightarrow x \in B(0,\sqrt{n})$, i.e. $\|x\|_2 \le \sqrt{n}$

$\{-1,1\}^n$: Boolean cube $\subset$ $B(0,\sqrt{n})$: Euclidean ball of radius $\sqrt{n}$

$$\max_{x\in\{\pm1\}^n} \sum_{ij} a_{ij} x_i x_j \le \max_{\|x\|_2 \le \sqrt{n}} \sum_{ij} a_{ij} x_i x_j = (\sqrt{n})^2 \max_{\|x\|_2 \le 1} \underbrace{\sum_{ij} a_{ij} x_i x_j}_{= \langle Ax, x\rangle}$$

If A is symmetric:

$$= n \cdot \underbrace{\|A\|}_{\text{"spectral norm"}} = n \cdot \underbrace{\lambda_1(A)}_{\text{largest eigenvalue of } A}$$

Maximizer x = the corresponding eigenvector.

- Similarly, $\exists$ Sp.R. for ①, ③.
- ☺ Sp.R is efficiently computable (diagonalize A)
- ☹ Tightness of SpR is unclear. No guarantees.

An alternative is:

SEMIDEFINITE RELAXATION

- Convex program: maximize a linear function over a convex set. (linear: generally, concave)
- Efficient solvers. Convex relaxation of ②:

$$\max_{x_i \in \{\pm 1\}} \sum_{ij} a_{ij} \underbrace{\boxed{x_i x_j}}_{=: Z_{ij}}$$

- Matrix $Z = [x_i x_j]_{i,j=1}^n = x x^T$

  (a) symmetric

  (b) $Z_{ii} = x_i^2 = 1 \quad \forall i$

  (c) positive semidefinite $(Z \succcurlyeq 0)$: $\quad u^T Z u \ge 0 \quad \forall u \in \mathbb{R}^n$

  $[\,u^T x x^T u = \langle u, x\rangle\langle x, u\rangle = \langle x, u\rangle^2 \ge 0\,]$

- Hence, relax:

$$\max_{x_i \in \{\pm 1\}} \sum_{i,j} a_{ij} x_i x_j \leq \max_{\substack{Z \succeq 0, \\ Z_{ii} = 1 \ \forall i}} \sum a_{ij} Z_{ij} = SDP(A)$$

- The objective function $\sum_{ij} a_{ij} Z_{ij} = \langle A, Z \rangle$ is linear in Z
- The feasible set $\{ Z : Z \succeq 0, Z_{ii} = 1 \ \forall i \}$ is convex

  $\{ Z : Z \succeq 0 \} \subset \mathbb{R}^{n^2}$ is convex "Spectrahedron"

  $\{ Z : Z_{ii} = 1 \ \forall i \}$ is a linear subspace $\Rightarrow$ convex

  Intersection of convex $\Rightarrow$ convex

- $\Rightarrow$ RHS of $(*)$ is a

  convex program. "Semidefinite program".