LECTURE 17

- Last class, we proposed a semidefinite relaxation:

$$\underset{x_i \in \{\pm 1\}}{\max} \sum_{i,j} a_{ij} \underbrace{x_i x_j}_{\substack{\shortparallel \\ Z_{ij}}} \leq \underset{\substack{Z \succeq 0, \\ Z_{ii}=1 \ \forall i}}{\max} \sum a_{ij} Z_{ij} \qquad (*)$$

combinatorial, NP hard — semidefinite program, tractable

- We will prove tightness: $RHS \leq c \cdot LHS$.
- But before that, let's rewrite $(*)$ in terms of:

- **Def/Prop** The Gram matrix of vectors $v_1, \dots, v_n \in \mathbb{R}^d$ is $[\langle v_i, v_j \rangle]_{i,j=1}^n$.
  It is always PSD. (DIY)
  Conversely, $\forall$ $n \times n$ PSD matrix is a Gram matrix of some vectors $v_1, \dots, v_n \in \mathbb{R}^n$. (DIY: consider the square root)

- In $RHS(*)$, $Z = [\langle v_i, v_j \rangle]$, $Z_{ii} = \langle v_i, v_i \rangle = \|v_i\|_2^2 = 1 \ \forall i \Rightarrow$

$$RHS(*) = \boxed{\underset{\substack{v_i \in \mathbb{R}^n \\ \text{unit}}}{\max} \sum_{ij} a_{ij} \langle v_i, v_j \rangle} \qquad (**)$$

SDP $\Rightarrow$ tractable (1000's of variables ok)

- We will work out tightness of Max Cut:

SEMIDEFINITE RELAXATION OF MAX CUT

[Goemans–Williamson '1994]

- Recall from Lec. 16 that for a graph G with adjacency matrix $[a_{ij}]$,

$$\text{MaxCut}(G) = \frac{1}{4} \max_{x_i \in \{\pm 1\}} \underbrace{\sum_{i,j=1}^{n} a_{ij}(1 - x_i x_j)}$$

$x_i = 1 \quad x_j = -1$

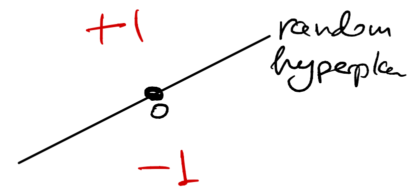
- According to $(*)$, a semidefinite relaxation is

$$\text{SDP}(G) = \frac{1}{4} \max_{\substack{v_i \in \mathbb{R}^n \\ \text{unit}}} \sum_{i,j=1}^{n} a_{ij} \left(1 - \langle v_i, v_j \rangle\right).$$

Trivially,

$$\text{MaxCut}(G) \le \text{SDP}(G) \qquad (v_i = (\pm 1, 0, 0, \dots)) \qquad (*)$$

- Solve SDP; How do we convert solution (v_i) to (x_i) ?



- "Randomized rounding"

Let $g \sim N(0, I_n)$. $\boxed{v \longmapsto \text{sign}\langle v, g \rangle}$

rotation invariant

- Proposed alg.:
  - (a) Solve SDP(G) $\Longrightarrow$ solution (v_i)
  - (b) Randomized rounding: $x_i := \text{sign}\langle v_i, g \rangle$.
  - Output partition: $i : x_i = 1 \quad j : x_j = -1$

Accuracy:

$\boxed{\text{\underline{THM} The expected \underline{cut} of this partition} \ge 0.878 \cdot \text{MaxCut}(G)}$

Proof Expected cut $= \mathbb{E}\, \frac{1}{4} \sum_{i,j=1}^{n} a_{ij}(1-x_i x_j)$

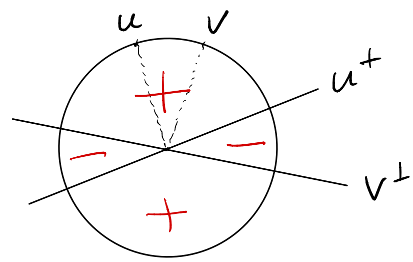
$$= \frac{1}{4} \sum_{i,j=1}^{n} a_{ij}\big(1-\underbrace{\mathbb{E}\, x_i x_j}_{\mathbb{E}\,\mathrm{sign}\langle v_i,g\rangle\langle v_j,g\rangle \,=\, ?}\big)$$

$=$

Lem (Grothendieck's identity) $\forall$ unit vectors $u,v \in \mathbb{R}^n$, $g \sim N(0,I_n)$:

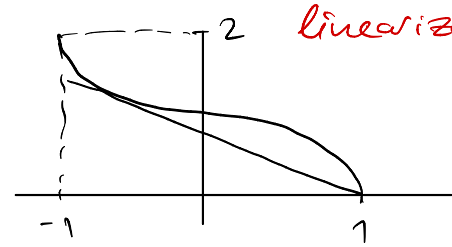
$$\boxed{\mathbb{E}\, \mathrm{sign}\langle u,g\rangle\, \mathrm{sign}\langle v,g\rangle = \frac{2}{\pi} \arcsin\langle u,v\rangle.}$$

Rotation invariance
$\Rightarrow$ WLOG, $u,v \in \mathbb{R}^2$
$\Rightarrow$ $g \sim N(0,I_2)$
$g \mapsto g/\|g\|_2 =: \theta \sim \mathrm{Unif}(\text{circle})$.
$\Rightarrow$ $\mathbb{E}\,\mathrm{sign}\langle u,\theta\rangle\,\mathrm{sign}\langle v,\theta\rangle$ = length ("+" arcs) − length ("−" arcs).
Then trigonometry (DIY).



$$= \frac{1}{4} \sum_{i,j=1}^{n} a_{ij}\left(1-\frac{2}{\pi}\arcsin\langle v_i,v_j\rangle\right) = \frac{1}{4}\sum_{ij} a_{ij}\cdot \underbrace{\frac{2}{\pi}\arccos\langle v_i,v_j\rangle}_{\text{linearize?}}$$

Fact: $\frac{2}{\pi}\arccos\theta \geq 0.878(1-\theta)$



$$\geq 0.878 \cdot \underbrace{\frac{1}{4}\sum_{ij} a_{ij}(1-\langle v_i,v_j\rangle)}_{\|}$$

$$\underset{(*)\,p.2}{\mathrm{SDP}(G)} \geq \mathrm{MaxCut}(G).$$

QED.