LECTURE 18

- Given numbers a_{ij}, $i=1,\dots,n$, $j=1,\dots,m$:

$$\underbrace{\max_{x_i, y_j \in \{\pm 1\}} \sum_{ij} a_{ij} x_i y_j}_{\substack{\text{combinatorial} \\ \text{NP-hard}}} \le \underbrace{\max_{\substack{u_i, v_j \in \mathbb{R}^d \\ \text{unit}}} \sum_{ij} a_{ij} \langle u_i, v_j \rangle}_{\substack{\text{semidefinite} \quad (\text{why?}) \\ \Rightarrow \text{tractable.}}}$$

relaxation

- Tightness:

THM (Grothendieck's inequality '1953) $\forall a_{ij}$, $\forall d$:

$$\boxed{\max_{\substack{u_i, v_j \in \mathbb{R}^d \\ \text{unit}}} \sum_{ij} a_{ij} \langle u_i, v_j \rangle \le 1.782 \cdot \max_{x_i, y_j \in \{\pm 1\}} \sum_{ij} a_{ij} x_i y_j}$$

Proof will be based on

(a) Probability (randomized rounding): Grothendieck identity (lec. 17).

$$\mathbb{E}\, \mathrm{sign}\langle u, g\rangle \, \mathrm{sign}\langle v, g\rangle = \frac{2}{\pi} \underbrace{\arcsin}\langle u, v\rangle \quad \forall \text{ unit } u, v.$$

(b) Kernel trick to handle the nonlinearity:

We will find maps $\phi, \psi : \mathbb{R}^n \to \mathbb{R}^N$ $(N \gg n)$ such that

$$\mathbb{E}\, \mathrm{sign}\langle \phi(u), g\rangle \, \mathrm{sign}\langle \psi(v), g\rangle = \beta \langle u, v\rangle$$

$$\| \text{ G.I}$$

$$\frac{2}{\pi} \arcsin \langle \phi(u), \psi(v)\rangle = \beta \langle u, v\rangle$$

$$\Updownarrow$$

$$\boxed{\langle \phi(u), \psi(v)\rangle = \sin\left(\frac{\beta\pi}{2}\langle u, v\rangle\right)} \qquad (*)$$

PLAN: We will build more and more nontrivial nonlinearities:

$$\langle u, v\rangle^2 \longrightarrow \langle u, v\rangle^k \longrightarrow \forall \text{ polynomial} \longrightarrow \forall \text{ analytic fn.}$$

TENSOR CALCULUS

① For a vector $u \in \mathbb{R}^n$, <u>tensor product</u>

$$u \otimes u := [u_i u_j]_{i,j=1}^n = uu^T \in \mathbb{R}^{n^2}$$

- $\langle u \otimes u, v \otimes v \rangle = \sum_{i,j} u_i u_j \cdot v_i v_j = \left(\sum u_i v_i\right)\left(\sum u_j v_j\right) = \langle u, v \rangle^2$

- $\Rightarrow$ $\exists$ map $\phi : \mathbb{R}^n \to \mathbb{R}^{n^2}$ such that

$$\langle \phi(u), \phi(v) \rangle = \langle u, v \rangle^2 \qquad (\phi(u) = u \otimes u)$$

unit vectors $\mapsto$ unit vectors.

SQUARE built.

② More generally,

$$u \otimes u \otimes u = [u_i u_j u_k]_{i,j,k=1}^n \in \mathbb{R}^{n^3}$$

and $\quad u^{\otimes k} \in \mathbb{R}^{n^k}$

$\Rightarrow$ $\exists$ map $\phi : \mathbb{R}^n \to \mathbb{R}^{n^2}$:

$$\langle \phi(u), \phi(v) \rangle = \langle u, v \rangle^k \qquad (\phi(u) = u^{\otimes k})$$

unit vectors $\to$ unit vectors.

$\forall$ MONOMIAL built.

③ <u>Direct sum</u> of $u \in \mathbb{R}^n$, $v \in \mathbb{R}^m$:

$$u \oplus v = [u_1, \dots, u_n, v_1, \dots, v_m] \in \mathbb{R}^{n+m}.$$

$$u \oplus v \oplus w = \text{etc.}$$

$$\langle u \oplus v, x \oplus y \rangle = \langle u, x \rangle + \langle v, y \rangle. \qquad (\oplus\times)$$

• <u>Ex</u> $\exists\ \phi : \mathbb{R}^n \to \mathbb{R}^N$: $\langle \phi(u), \phi(v) \rangle = 2\langle u,v \rangle + 3\langle u,v \rangle^2 + 5\langle u,v \rangle^3$.

$\ulcorner \phi(u) = (\sqrt{2}\,u) \oplus (\sqrt{3}\,u \otimes u) \oplus (\sqrt{5}\,u \otimes u \otimes u) \in \mathbb{R}^{n+n^2+n^3} \lrcorner$

$\Rightarrow$ $\forall$ <u>polynomial with nonnegative coeffs</u> built.

④ If some coeff's are <u>negative</u>, impossible ($\langle \phi(u), \phi(u) \rangle \geq 0$)
but:

• <u>Ex</u> $\exists\ \phi, \psi : \mathbb{R}^n \to \mathbb{R}^N$: $\langle \phi(u), \psi(v) \rangle = 2\langle u,v \rangle - 3\langle u,v \rangle^2 - 5\langle u,v \rangle^3$.

$\ulcorner$ $\phi(u)$ as above;
$\psi(u) := (\sqrt{2}\,u) \oplus (-\sqrt{3}\,u \otimes u) \oplus (-\sqrt{5}\,u \otimes u \otimes u)$. $\lrcorner$

$\Rightarrow$ $\forall$ <u>polynomial</u> built

<u>Remark</u>: $\forall$ <u>unit</u> u: $\quad \|\phi(u)\|_2^2 = \|\psi(u)\|_2^2 = 2+3+5 = 10.$

(5) Take limits $\Rightarrow$ $\forall$ real analytic function built:

Prop $\forall$ real analytic function $f(x) = \sum_k c_k x^k$, $\forall \varepsilon > 0$

$\exists\ \phi, \psi : \mathbb{R}^n \to \ell^2$ (define it!)

$$\langle \phi(u), \psi(v) \rangle = f(\langle u, v \rangle) \quad \forall u, v \in \mathbb{R}^n.$$

Moreover,
$$\|\phi(u)\|_2^2 = \|\psi(u)\|_2^2 = \sum_k |c_k|$$

Now we are ready to prove (* p.1):

Lem $\exists\ \phi, \psi : \mathbb{R}^n \to \ell^2$ that take unit vectors to unit vectors,

and
$$\langle \phi(u), \psi(v) \rangle = \sin\Big(\underbrace{\tfrac{\beta\pi}{2}}_{=:c} \langle u, v \rangle\Big) \quad \forall\ u, v \text{ unit}$$

where $\beta = \frac{2}{\pi} \ln(1+\sqrt{2})$.

Apply Prop. for

$$f(x) = \sin(cx) = cx - \frac{(cx)^3}{3!} + \frac{(cx)^5}{5!} - \frac{(cx)^7}{7!} + \dots$$

To make sure ϕ maps unit vectors to unit vectors, set

$$1 = \sum_k |c_k| = c + \frac{c^3}{3!} + \frac{c^5}{5!} + \frac{c^7}{7!} + \dots = \frac{e^c - e^{-c}}{2}$$

Solve $\Rightarrow$ $c = \ln(1+\sqrt{2}) =: \frac{\beta\pi}{2}$ $\Rightarrow$ $\beta = \frac{2}{\pi} \ln(1+\sqrt{2})$.

Hence (see p.1):
$$\boxed{\mathbb{E}\, \mathrm{sign}\langle \phi(u), g \rangle\, \mathrm{sign}\langle \psi(v), g \rangle = \beta \langle u, v \rangle} \quad (*)$$

Proof of G. Inequality (Krivine's argument '1979) $\forall u_i, v_j$ unit:

let $u_i' := \phi(u_i) \in \ell^2$, $v_j' := \psi(v_j) \in \ell^2$; $i=1,\dots,n$; $j=1,\dots,m$.

W.L.O.G. we can assume that $u_i, v_j \in \mathbb{R}^{n+m}$ (explain).

$$\mathbb{E}\,\mathrm{sign}\langle u_i', g\rangle \,\mathrm{sign}\langle v_j', g\rangle \overset{G.1}{=\!=} \arcsin\left(\frac{2}{\pi}\langle u_i', v_j'\rangle\right) \overset{\text{Prop}}{=\!=} \beta\langle u_i, v_j\rangle.$$

$$\sum_{ij} a_{ij}\langle u_i, v_j\rangle = \frac{1}{\beta}\sum_{ij} a_{ij}\, \mathbb{E}\, \underbrace{\mathrm{sign}\langle u_i', g\rangle}_{= x_i}\, \underbrace{\mathrm{sign}\langle v_j', g\rangle}_{= y_j} \qquad (* \text{ p.3})$$

$$= \frac{1}{\beta}\,\mathbb{E}\left[\sum_{ij} a_{ij} x_i y_j\right]$$

$$\leq \boxed{\frac{1}{\beta}} \max_{x_i, y_j \in \{\pm 1\}} \sum_{ij} a_i x_j$$

$$\frac{1}{\beta} = \frac{\pi}{2\ln(1+\sqrt{2})} = 1.7822. \qquad \text{QED}.$$

QUESTION

- Is Krivine's constant optimal?

NO [Braverman et al. '2011]