LECTURE 28

- Work with matrices as with numbers?

  random matrices $\sim$ random variables?

  $\leftarrow$ CLT, LLN, concentration

FUNCTIONAL CALCULUS (5.4)

Symmetric $d \times d$ matrices.

- Adding, subtracting, multiplying, dividing $(A/B = AB^{-1})$
- 0; $1 \mapsto I$
- $|a| \to \|A\|$ (operator norm)
- Functions of matrices: if $A = \sum_{i=1}^{d} \lambda_i u_i u_i^T$ (spectral decomposition)

then $$A^2 = \sum_{i=1}^{d} \lambda_i^2 u_i u_i^T, \qquad A^{-1} = \sum_{i=1}^{d} \frac{1}{\lambda_i} u_i u_i^T,$$

$$I + A + A^2 = \sum_{i=1}^{d} (1 + \lambda_i + \lambda_i^2) u_i u_i^T$$

Similarly $\forall$ polynomial: $p(A) = \sum_{i=1}^{d} p(\lambda_i) u_i u_i^T \implies$

- $$\boxed{\text{Def} \quad \text{If } f: \mathbb{R} \to \mathbb{R} \text{ and } A = \sum_{i=1}^{d} \lambda_i u_i u_i^T \implies f(A) := \sum_{i=1}^{d} f(\lambda_i) u_i u_i^T}$$

Ex $$e^A = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \dots$$

$$\sum_{i=1}^{d} e^{\lambda_i} u_i u_i^T = \sum_{i=1}^{d} 1 \cdot u_i u_i^T + \sum_{i=1}^{d} \lambda_i u_i u_i^T + \frac{1}{2!} \sum_{i=1}^{d} \lambda_i^2 u_i u_i^T + \dots$$

$$e^{\lambda_i} = 1 + \lambda_i + \frac{1}{2!}\lambda_i^2 + \dots$$

$$= I + A + \frac{1}{2} A^2 + \dots$$

LÖWNER ORDER

on the set of $d \times d$ symmetric matrices.

Recall: A is PSD $\overset{def}{\iff} u^T A u \geq 0 \ \forall u \in \mathbb{R}^d \iff \lambda_i(A) \geq 0 \ \forall i$.

<u>Def</u> (<u>Löwner order</u>) We say that $A \succcurlyeq 0$ if A is PSD.

We say $A \succcurlyeq B$ and $B \preccurlyeq A$ if $A - B \succcurlyeq 0$

i.e. $u^T A u \geq u^T B u \ \forall u \in \mathbb{R}^d$.

- $\preccurlyeq$ is a <u>partial</u> order, but not a <u>total</u> order: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is neither $\succcurlyeq 0$ nor $\preccurlyeq 0$.

SKIP PROOF

- Reflexivity: $X \preccurlyeq X$ is obvious ($0 \preccurlyeq 0$)
- Antisymmetry: $\underline{X \preccurlyeq Y \text{ and } Y \preccurlyeq X} \overset{?}{\Rightarrow} X = Y$

$$\Downarrow$$
$$u^T X u = u^T Y u \quad \forall u \in \mathbb{R}^d$$
$$\Downarrow$$
$$u^T (X - Y) u = 0 \quad \forall u \quad \Rightarrow \underbrace{\|X - Y\|}_{\text{operator norm}} = \max_{u \text{ unit}} u^T (X - Y) u = 0$$
$$\Rightarrow X = Y \quad ✓$$

- Transitivity: $X \preccurlyeq Y$ and $Y \preccurlyeq Z \overset{?}{\Rightarrow} X \preccurlyeq Z$

($u^T X u \leq u^T Y u \leq u^T Z u$. ✓)

(Ex)

- Matrix multiplication is <u>not commutative</u> ($XY \neq YX$).

$\Rightarrow$ some basic facts don't transfer from scalars to matrices.

- For example, the implication

$$\text{“} X \succcurlyeq 0, \ Y \succcurlyeq 0 \quad \Rightarrow \quad XY \succcurlyeq 0 \text{”}$$

holds for commuting X, Y

Check! X, Y are diagonalizable by the same orthogonal matrix

but in general XY needs not even be <u>symmetric</u>!

- A way around: multiply on <u>both</u> sides:

<u>Prop</u> If $X \succcurlyeq Y$ and A is any $d \times d$ $\Rightarrow$ $A^T X A \succcurlyeq A^T Y A$

WLOG $Y = 0$

$$u^T A^T X A u = (Au)^T X (Au) \geq 0 \quad \forall u$$

Eigenvalues & Löwner order

- Recall Courant-Fisher min-max theorem:

$$\lambda_i(X) = \max_{\dim E = i} \min_{\substack{u \in E \\ \text{unit}}} u^T X u$$

$$\Downarrow \qquad\qquad \rotatebox{90}{$\leq$} \quad u^T Y u \text{ if } X \preceq Y$$

- $\boxed{\textbf{Prop 3} \text{ (Eigenvalue monotonicity) } \quad X \preceq Y \Rightarrow \lambda_i(X) \leq \lambda_i(Y) \ \forall i}$

- Remark: The converse holds for commuting matrices but not in general. **(HW)**

Cor (Trace monotonicity): Let $f: \mathbb{R} \to \mathbb{R}$ be an increasing function. Then

$$X \preceq Y \Rightarrow \operatorname{tr} f(X) \leq \operatorname{tr} f(Y)$$

$$\Big[\ \sum_{i=1}^{d} f(\lambda_i(X)) \leq \sum_{i=1}^{d} f(\lambda_i(Y)) \text{ by Prop. \& increasing } f \ \Big]$$

Ⓠ: Does the stronger result hold: "$X \preceq Y \Rightarrow f(X) \preceq f(Y)$" ?

Ans: It does for some f ("matrix monotone") but not all f **(HW)**

$\uparrow$
e.g. x^2

MATRIX MONOTONE FUNCTIONS

Prop 1 ($1/x$ is matrix monotone)

If X is invertible and $0 \preceq X \preceq Y$
then Y is invertible and $X^{-1} \succeq Y^{-1} \succeq 0$

1. X is invertible $\Rightarrow \lambda_i(X) > 0 \;\forall i \xrightarrow{\text{Prop. p. 3}} \lambda_i(Y) > 0 \;\forall i \Rightarrow Y$ invertible.

2. In the special case where $Y = I$: $0 \preceq B \preceq I \overset{?}{\Rightarrow} B^{-1} \succeq I$

$\Updownarrow \qquad \Updownarrow$

Since B, I commute, $0 \leq \lambda_i(B) \leq 1 \;\forall i \Rightarrow \frac{1}{\lambda_i(B)} \geq 1$.

3. In the general case: assume $0 \prec X \preceq Y$. Multiply on both sides by $Y^{-1/2}$ (Prop. p. 2) $\Rightarrow$
$0 \preceq Y^{-1/2} X Y^{-1/2} \preceq I$. Use part 2 $\Rightarrow$
$(Y^{-1/2} X Y^{-1/2})^{-1} = Y^{1/2} X^{-1} Y^{1/2} \succeq I$. Multiply on both sides by $Y^{-1/2}$ again $\Rightarrow$
$X^{-1} \succeq Y^{-1}$.

Prop 2 ($\log x$ is matrix monotone)

If X is invertible and $0 \preceq X \preceq Y$, then $\log X \preceq \log Y$.

Proof is based on scalar equation

$$\log x = \int_0^\infty \left(\frac{1}{1+t} - \frac{1}{x+t}\right) dt \qquad \forall x > 0 \qquad \text{(Check!)}$$

- Its matrix form is: $$\log X = \int_0^\infty \left[(1+t)^{-1} I - (X+tI)^{-1}\right] dt \qquad (*)$$

$\log \lambda_i = \int_0^\infty \left(\frac{1}{1+t} - \frac{1}{\lambda_i + t}\right) dt \Rightarrow$

$$\Rightarrow \log X = \sum_{i=1}^d \log(\lambda_i)\, u_i u_i^T = \int_0^\infty \sum_{i=1}^d \left(\frac{1}{1+t} - \frac{1}{\lambda_i+t}\right) u_i u_i^T \, dt$$

$$= \int_0^\infty \Bigg[\frac{1}{1+t} \underbrace{\sum_{i=1}^d u_i u_i^T}_{= I} - \underbrace{\sum_{i=1}^d \frac{1}{\lambda_i+t} u_i u_i^T}_{= (X+tI)^{-1}}\Bigg] dt.$$

- $\therefore X \preceq Y \Rightarrow X + tI \preceq Y + tI \xrightarrow{\text{Prop 1}} -(X+tI)^{-1} \preceq -(Y+tI)^{-1} \overset{(*)}{\Rightarrow} \log X \preceq \log Y$.

Remark General theory of matrix monotone functions: [K. Löwner '1936]