LECTURE 39

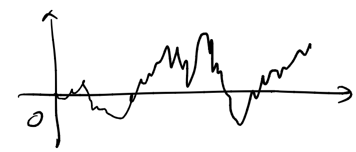
STOCHASTIC PROCESSES

- Consider $\forall$ Gaussian process $(X_t)_{t\in T}$ indexed by $\forall$ set T.

  Induces (pseudo-) metric on T:

$$\boxed{d(t,s) := \|X_t - X_s\|_{L^2}} = \left(\mathbb{E}(X_t - X_s)^2\right)^{1/2}$$

Ex (a) Brownian motion $(B_t)_{t\in[0,\infty)}$



$$B_t - B_s \sim N(0, |t-s|)$$

$$\Rightarrow \quad d(t,s) = \sqrt{|t-s|} \quad \text{on } \mathbb{R}.$$

(b) Brownian Bridge, fractional B.M., Ornstein-Uhlenbeck process, Brownian sheet, Gaussian free field, etc.

(c) $X_t := gt$ where $g \sim N(0,1)$, $t \in \mathbb{R} =: T$

$$\Rightarrow \quad X_t - X_s = g(t-s) \sim N\left(0, (t-s)^2\right)$$

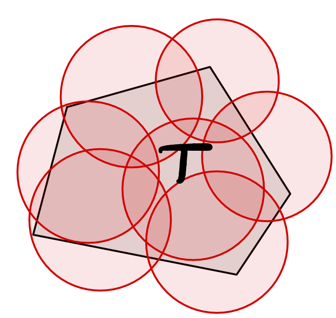
$$\Rightarrow \quad d(t,s) = |t-s| \quad \text{on } \mathbb{R}$$

(d) High-dimensional: $X_t := \langle g, t\rangle$ where $g \sim N(0, I_d)$, $t \in \mathbb{R}^d =: T$

$$\Rightarrow \quad d(t,s) = \|t-s\|_2 \quad \text{on } \mathbb{R}^d \qquad \text{(check!)}$$

- Goal: compute $\boxed{\mathbb{E}\sup_{t\in T} X_t}$ via geometry of the metric space (T,d),

  specifically the covering numbers



$N(T,d,\varepsilon)$ = min. cardinality of an ε-net of (T,d)

= min # of ε-balls that cover T

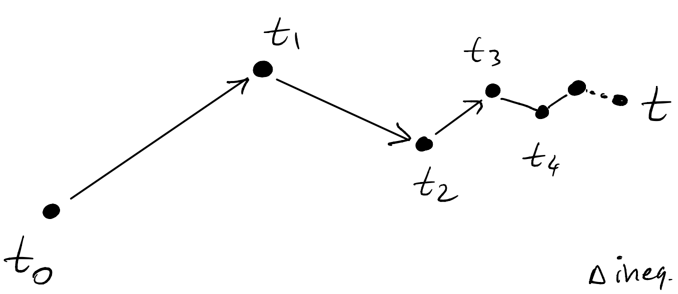
THM [Dudley's Integral Inequality 1967]

$\forall$ mean zero Gaussian process $(X_t)_{t\in T}$ satisfies

$$\mathbb{E}\sup_{t\in T} X_t \le 8\int_0^\infty \underbrace{\sqrt{\log N(T,d,\varepsilon)}}_{\text{metric entropy of } T}\, d\varepsilon.$$

<u>Proof</u>: "**CHAINING**" = Multiscale union bound.

- Set the scale
$$\varepsilon_k := 2^{-k}, \qquad k=0,1,2,\dots$$
- Choose ε-nets T_k of T so that
$$|T_k| = N(T,d,\varepsilon_k).$$
WLOG $\mathrm{diam}(T)\le 1 \Rightarrow$
$$T_0=\{t_0\}$$
- Def of ε-net $\Rightarrow$ $\forall\, t\in T \quad \exists \quad t_k\in T_k$:
$$d(t_k,t)\le\varepsilon_k \qquad (*)$$
- <u>Chaining</u>: progressive approximation of t:



$$X_{t_0}-X_t = (X_{t_0}-X_{t_1})+(X_{t_1}-X_{t_2})+\cdots$$

(WLOG T is finite $\Rightarrow$ sum is finite)

$$\overset{\Delta\text{ ineq.}}{\Longrightarrow} \quad |X_{t_0}-X_t| \le \sum_{k=1}^{\infty} |X_{t_{k-1}}-X_{t_k}|$$

- Note: $d(t_{k-1}, t_k) \overset{\Delta \text{ ineq.}}{\leq} d(t_{k-1}, t) + d(t, t_k) \overset{(*)}{\leq} \underset{\underset{2^{-k+1}}{\parallel}}{\varepsilon_{k-1}} + \underset{\underset{2^{-k}}{\parallel}}{\varepsilon_k} = 3\varepsilon_k$

$$\Rightarrow \max_{t \in T} |X_{t_0} - X_t| \leq \sum_{k=1}^{\infty} \max_{\substack{u \in T_{k-1} \\ v \in T_k \\ d(u,v) \leq 3\varepsilon_k}} |X_u - X_v| \qquad \left(\substack{u \in T_{k-1} \\ v \in T_k \\ d(u,v) \leq 3\varepsilon_k}\right) =: P_k$$

$$\Rightarrow \mathbb{E} \max_{t \in T} |X_{t_0} - X_t| \leq \sum_{k=1}^{\infty} \mathbb{E} \max_{(u,v) \in P_k} |X_u - X_v| .$$

- By assumption & def. of d (p.1), $X_u - X_v \sim N(0, d(u,v)^2)$ where $d(u,v) \leq 3\varepsilon_k$

Use

HW5, Problem 4 "Maximum of subgaussians": $\forall$ normal r.v.'s Z_i with mean 0, variance $\leq \sigma$: $\mathbb{E} \max_{i \leq n} \lvert Z_i \rvert \leq C\sigma\sqrt{\log n}$.

$$\Rightarrow \mathbb{E} \max_{(u,v) \in P_k} |X_u - X_v| \underset{\substack{\uparrow \\ \text{hides an absolute} \\ \text{constant factor}}}{\lesssim} \varepsilon_k \sqrt{\log \underbrace{|P_k|}_{\substack{\wedge\!\!\!| \\ |T_{k-1}| \cdot |T_k| \leq |T_k|^2}}} \lesssim \varepsilon_k \sqrt{\log |T_k|} .$$

- $$\Rightarrow \mathbb{E} \max_{t \in T} |X_{t_0} - X_t| \lesssim \sum_{k=1}^{\infty} \varepsilon_k \sqrt{\log |T_k|} = \sum_{k=1}^{\infty} \underbrace{2^{-k}}_{\substack{/\!/ \\ \int_{2^{-k-1}}^{2^{-k}} d\varepsilon}} \underbrace{\sqrt{\log N(T, d, \underbrace{\varepsilon^{-k}}_{\substack{\vee\!\!| \\ \varepsilon}})}}_{\substack{\wedge\!\!| \\ \log N(T,d,\varepsilon)}}$$

$$\leq \int_0^{\infty} \sqrt{\log N(T, d, \varepsilon)}\, d\varepsilon. \quad ☺$$

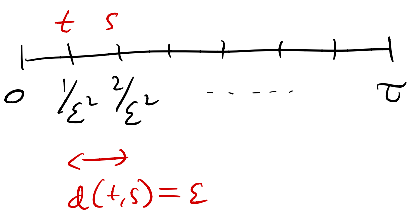
- Getting rid of X_{t_0}:

$$\sup_{t \in T} |X_t| = \sup_{t \in T} |X_t - \overbrace{\mathbb{E} X_{t_0}}^{\substack{0 \\ \parallel}}| \leq \sup_{t \in T} |X_t - X_{t_0}|$$

Jensen

Take expectation $\Rightarrow$ QED.

Ex Brownian motion $(B_t)_{t \in \overbrace{[0,\tau]}^{= T}}$ $\qquad d(t,s) = \sqrt{|t-s|}$ $\qquad \operatorname{diam}(T) = \sqrt{\tau}$



is an ε-net of card. τ/ε^2

$$\Rightarrow N(T,d,\varepsilon) \le \tau/\varepsilon^2 \quad \forall \varepsilon \le \sqrt{\tau}$$

and $= 1$ if $\varepsilon > \sqrt{\tau}$

$$\Rightarrow \int_0^\infty \sqrt{\log N(T,d,\varepsilon)}\, d\varepsilon = \int_0^{\sqrt{\tau}} \sqrt{\log(\tau/\varepsilon^2)}\, d\varepsilon \overset{\varepsilon = x\sqrt{\tau}}{=\!=} \sqrt{\tau} \int_0^1 \sqrt{\log(1/x^2)}\, dx \le C\sqrt{\tau}$$

$$\Rightarrow \mathbb{E} \sup_{t \in [0,\tau]} B_t \le C\sqrt{\tau}.$$

This is optimal. In fact, $\underbrace{\sup_{t\in[0,\tau]} B_t}_{} \overset{\text{dist}}{=} |N(0,\tau)|$ (a consequence of the "reflection principle")

$$\Rightarrow \quad \mathbb{E} \sup_{t \in [0,\tau]} B_t = \sqrt{\frac{2\tau}{\pi}}.$$

- **Remark**: proof works $\forall$ subgaussian process, yields:

> **THM** (Subgaussian Dudley)
>
> Let $(X_t)_{t \in T}$ be a mean zero stochastic process on a metric space (T,d), which satisfies
>
> $$\|X_t - X_s\|_{\psi_2} \le K \cdot d(t,s) \qquad \forall\, t,s \in T$$
>
> Then
>
> $$\mathbb{E} \sup_{t \in T} X_t \le CK \int_0^\infty \sqrt{\log N(T,d,\varepsilon)}\, d\varepsilon.$$