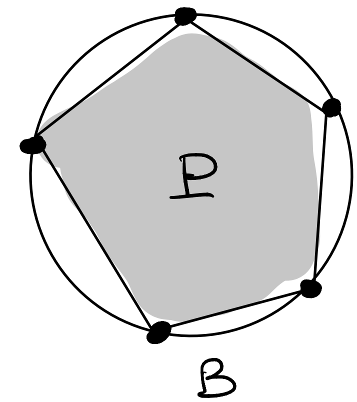
LECTURE 4

THM (Volume of polytopes) [Carl–Pajor '88]

Let B = unit Euclidean ball, $P \subset B$ polytope with m vertices. Then

$$\frac{\mathrm{Vol}(P)}{\mathrm{Vol}(B)} \le \left(4\sqrt{\frac{\log m}{n}}\right)^n.$$



exponentially small unless $m > \exp(cn)$ vertices.

Proof. By def. of covering numbers, P can be covered by $N(P,\varepsilon)$ copies of a ball of radius ε, denoted εB. Comparing the volumes yields

$$\mathrm{Vol}(P) \le N(P,\varepsilon) \cdot \mathrm{Vol}(\varepsilon B).$$

diam $(P) \le$ diam $(B) = 2$

- By Thm p. 4 (lec. 3), $N(P,\varepsilon) \le m^{\frac{\mathrm{diam}(P)^2}{2\varepsilon^2}} \le m^{2/\varepsilon^2}$
- By volume scaling, $\mathrm{Vol}(\varepsilon B) = \varepsilon^n \cdot \mathrm{Vol}(B)$.

Substitute $\Rightarrow$

$$\frac{\mathrm{Vol}(P)}{\mathrm{Vol}(B)} \le m^{2/\varepsilon^2} \cdot \varepsilon^n$$

- Minimize RHS in ε $\Rightarrow$ QED. (DIY: take logs & differentiate)

Remarks

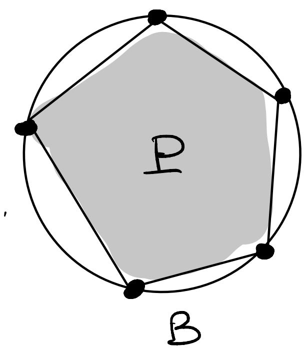
① [Carl-Pajor] proved a slightly sharper bound $\left(C\sqrt{\frac{\log(m/n)}{n}}\right)^n$.

② It is optimal, attained by a random polytope

$$P = \mathrm{conv}\{x_1, \dots, x_m\}$$

where x_i = indep. random pts on the unit sphere.

[Dafnis-Giannopoulos-Tsolomitis '2003, 2009]



correction of intuition

③ "Hyperbolic intuition"

for $\delta := \sqrt{\frac{4\log m}{n}}$, Thm says:

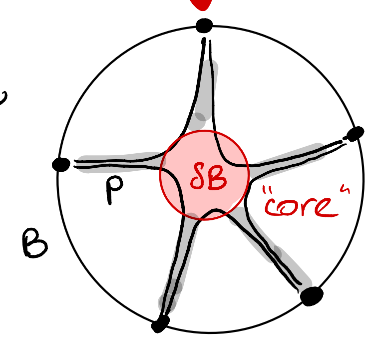
tiny ball!

$$\mathrm{Vol}(P) \le \delta^n \,\mathrm{Vol}(B) = \mathrm{Vol}(\delta B).$$

Thus the picture "actually" looks like this,

even though it violates convexity.

Milman's "hyperbolic intuition"



④ A "blind spider" experiment:

- A random walk will stay in the "core" δB, will not find the "tentacles".

- The spider can't tell the polytope from the ball δB.

- Bad for algorithms (MCMC)

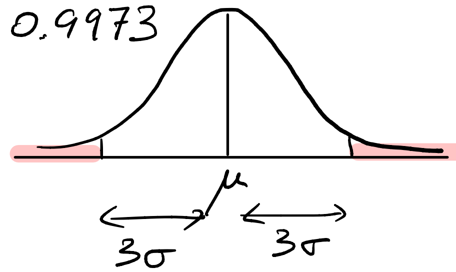
CONCENTRATION INEQUALITIES.

- "$X \approx \mathbb{E}X$ with high probability"

closer to 1 than you think.

- Ex: normal distribution. $X \sim N(\mu, \sigma^2)$ satisfies

$$|X - \mu| \le 3\sigma \quad \text{with prob. } 0.9973$$



(see 68-95-99.7 rule)

A general tail bound:

Prop (Gaussian tails) $g \sim N(0,1)$ satisfies

$$\mathbb{P}\{g \ge t\} \le \frac{1}{t\sqrt{2\pi}} e^{-t^2/2} \qquad \forall t \ge 0$$

decays fast in t.

Proof (Recall: $\mathbb{P}\{X \in A\} = \int_A p(x)\,dx$ density of X)

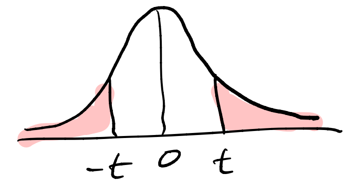
$$\mathbb{P}\{g \ge t\} = \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx. \qquad \text{Not computable analytically.}$$

To estimate, change variables $x = t + y \Rightarrow \frac{x^2}{2} = \frac{t^2}{2} + ty + \frac{y^2}{2}$

$$\mathbb{P}\{g \ge t\} = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} e^{-ty} \underbrace{e^{-y^2/2}}_{\le 1} dy$$

$$\le \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \underbrace{\int_0^\infty e^{-ty}\,dy}_{= 1/t \text{ (DIY)}}. \qquad \text{QED.}$$

- By symmetry,



$$\mathbb{P}\{|g| \geq t\} = 2 \cdot \mathbb{P}\{g \geq t\} \leq \frac{1}{t}\sqrt{\frac{2}{\pi}}\, e^{-t^2/2} \qquad (*)$$

- More generally, if $X \sim N(\mu, \sigma^2) \Rightarrow X = \mu + \sigma g$

$$\Rightarrow \mathbb{P}\{|X - \mu| \geq t\sigma\} = \mathbb{P}\{|g| \geq t\} \leq e^{-t^2/2} \qquad \forall\, t \geq 1.$$

- <u>Ex</u>: $t = 3$: $\quad \mathbb{P}\{|X - \mu| \leq 3\sigma\} \geq 1 - \frac{1}{3}\sqrt{\frac{2}{\pi}}\, e^{-3^2/2} \geq 0.997$

almost as good as the exact number 0.9973 on p. 3 ☺

- <u>Remark</u>: a simpler form of (*) often suffices.
  Since $\sqrt{2/\pi} \leq 1$, we have:

$$\boxed{\mathbb{P}\{|g| \geq t\} \leq e^{-t^2/2} \qquad \forall t \geq 1}$$

"Gaussian tail bound"