LECTURE 5

- For general distributions, Chebyshev's inequality:

- **Prop** (Markov's inequality) $\forall$ non-negative r.v. X:

$$\boxed{\mathbb{P}\{X \geq t\} \leq \frac{\mathbb{E}X}{t} \qquad \forall t > 0.}$$

Proof $\forall x \in \mathbb{R}$ can be decomposed as

$$x = x \cdot \mathbb{1}_{\{x \geq t\}} + x \cdot \mathbb{1}_{\{x < t\}}$$

($\mathbb{1}_A$ is the indicator: $\mathbb{1}_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if not} \end{cases}$)

Apply this for X and take expectations on both sides:

$$\mathbb{E}X = \mathbb{E}[\underbrace{X}_{\geq t} \mathbb{1}_{\{X \geq t\}}] + \underbrace{\mathbb{E}[X \mathbb{1}_{\{X < t\}}]}_{\geq 0}$$

$$\geq t\, \mathbb{E}\, \mathbb{1}_{\{X \geq t\}} = t \cdot \mathbb{P}\{X \geq t\}.$$

Divide both sides by $t \Rightarrow$ QED

- **Prop** (Chebyshev's inequality) $\forall$ r.v. X with mean μ, variance σ^2:

$$\boxed{\mathbb{P}\{|X-\mu| \geq t\} \leq \frac{\sigma^2}{t^2} \qquad \forall t > 0}$$

Proof $\mathbb{P}\{|X-\mu| \geq t\} = \mathbb{P}\{(X-\mu)^2 \geq t^2\} \leq \frac{\mathbb{E}(X-\mu)^2}{t^2}$ (Markov for $(X-\mu)^2$)

$$= \sigma^2 / t^2.$$

QED

- QUESTION: Toss a fair coin N times. $P\{\text{at least } \frac{3}{4}N \text{ heads}\} = ?$

- Solution 1, based on Chebyshev:

$$S_N = \#\text{heads} \sim \text{Binom}(N, \tfrac{1}{2})$$

$$\mathbb{E} S_N = \frac{N}{2}, \quad \text{Var}(S_N) = \frac{N}{4}$$

Chebyshev $\Rightarrow$

$$P\{S_N \geq \tfrac{3}{4}N\} = \mathbb{P}\left\{ |S_N - \tfrac{N}{2}| \geq \tfrac{N}{4} \right\} \leq \frac{N/4}{(N/4)^2} = \boxed{\frac{4}{N}}$$

$= 0.05$ if $N = 80$

- Solution 2, based on CLT:

$$\frac{S_N - \mathbb{E} S_N}{\sqrt{\text{Var}(S_N)}} \to N(0,1) \quad \text{as} \quad N \to \infty$$

$$\Rightarrow \mathbb{P}\{S_N \geq \tfrac{3}{4}N\} = \mathbb{P}\left\{ \frac{S_N - N/2}{\sqrt{N/4}} \geq \sqrt{\frac{N}{4}} \right\}$$

$$\approx \mathbb{P}\{g \geq \underbrace{\sqrt{\tfrac{N}{4}}}_{=:t}\} \quad \text{where } g \sim N(0,1)$$

$$\leq e^{-t^2/2} = \boxed{e^{-N/8}} \quad \text{(Gaussian tail: lec. 4 p.4)}$$

≈ 0.000045 if $N = 8$

MUCH BETTER!

- But Sol. 2 has a gap: the error term in CLT.

What is it?

$1/\sqrt{N}$!

• QUANTITATIVE CLT:

THM (Berry–Esseen) Let X_i be iid r.v's with mean 0, var. 1,

$$\Rightarrow \left| P\left\{ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i \ge t \right\} - P\{g \ge t\} \right| \le \frac{\rho}{\sqrt{N}}$$

where $g \sim N(0,1)$ and $\rho = E|X_1|^3$.

• THAT'S SAD : taking this error into account in Sol. 2 yields probability

$$\frac{1}{\sqrt{N}} + e^{-N/8}$$

↑ BIG

Not better than Sol. 1 based on Chebyshev.

• Can we improve $\frac{1}{\sqrt{N}}$ in CLT?

NO : $P\{\text{exactly } \frac{N}{2} \text{ heads}\} = P\{S_N = \frac{N}{2}\} = 2^{-N}\binom{N}{N/2} \asymp \frac{1}{\sqrt{N}}$

while $P\{g=0\} = 0$

$\Downarrow$

error $\frac{1}{\sqrt{N}}$ is unavoidable.

WHAT SHOULD WE DO?

Sidestep CLT, prove a direct bound:

THM (Hoeffding's inequality) Let $X_1, \dots, X_N$ be independent symmetric Bernoulli r.v's, i.e. $P\{X_i = 1\} = P\{X_i = -1\} = 1/2$.

Then
$$P\left\{ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i \geq t \right\} \leq \exp\left(-\frac{t^2}{2}\right) \qquad \forall t \geq 0.$$

Gaussian tail! ☺

- Apply HI for the coin flip problem $\Rightarrow$
  $P\{\text{at least } \frac{3}{4}N \text{ heads}\} \leq e^{-N/8}$ (Ex). ☺

- Proof: next class.