LECTURE 8

REMARKS on Chernoff's inequality:

① Lower tails are similar:
$$\boxed{\mathbb{P}\{S_N \le t\} \le e^{-\mu}\left(\frac{e\mu}{t}\right)^t \quad \forall\, 0<t<\mu}$$
(HW Problem)

② Chernoff is optimal: if $S_N \sim \mathrm{Binom}(N, \mu)$ then
$$\boxed{\mathbb{P}\{S_N \ge t\} \ge (\mu/t)^t} \quad \forall\, 1 \le t \le N.$$

② Large deviations: when t is large, the "Poisson tail" $\sim t^{-t} = e^{-t\log t}$ is heavier than Gaussian $e^{-t^2/2}$.

③ Small deviations: when $t \approx \mu$, say $t = (1+\underset{\substack{\uparrow\\ \text{small}}}{\delta})\mu$,

- $$e^{-\mu}\left(\frac{e\mu}{t}\right)^t = e^{-\mu}\left(\frac{e}{1+\delta}\right)^{(1+\delta)\mu} = e^{\delta\mu}\left(\frac{1}{1+\delta}\right)^{(1+\delta)\mu}$$
$$= \exp\Big[\mu\big(\underbrace{\delta - (1+\delta)\log(1+\delta)}_{\le\, -\delta^2/6 \text{ if}}\big)\Big] \le \exp\left(-\delta^2\mu/6\right)$$

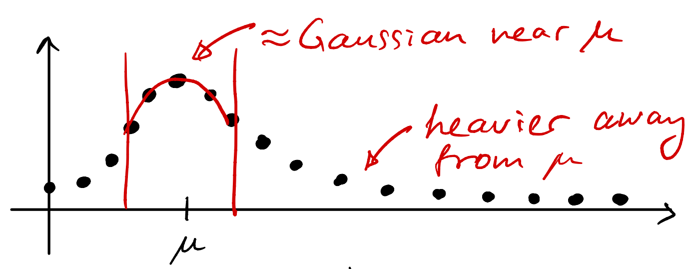
GAUSSIAN TAIL! ☺

(DIY by Taylor)

- Combine upper & lower tails

$\Downarrow$

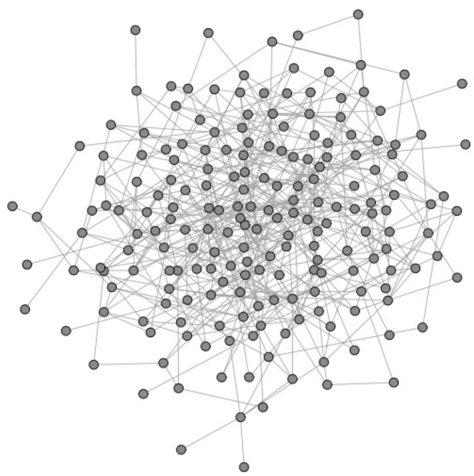
Cor (Chernoff's ineq: small deviations)
$$\boxed{\mathbb{P}\{|S_N - \mu| \ge \delta\mu\} \le \exp(-\delta^2\mu/6) \quad \forall\, \delta \in [0,1]}$$



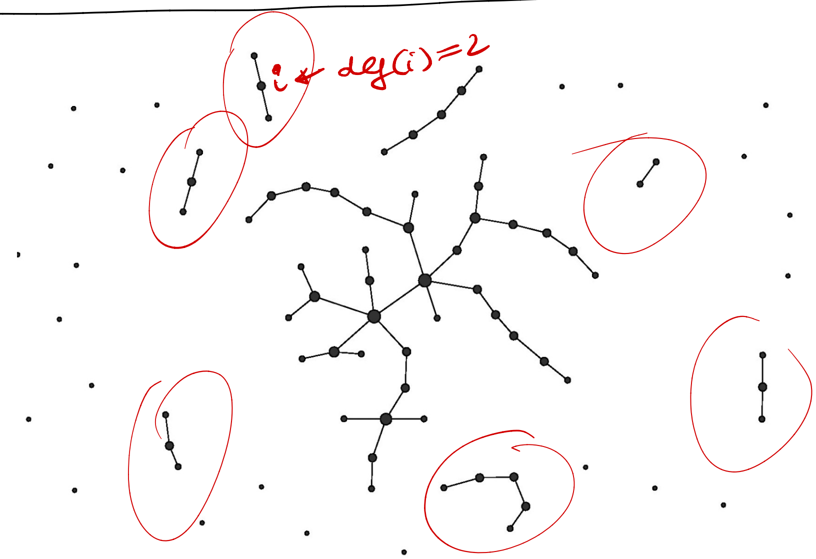
APPLICATION: RANDOM GRAPHS

Def Erdös-Renyi model $G(N,p)$:

> Fix a set of N vertices.
> Connect each pair of vertices with an edge independently, with prob. p.



$G(N,p)$ for $N=200$, $p=\frac{1}{40}$



$G(N,p)$ for $p=\frac{1}{100}$.

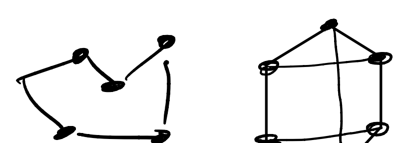
- **Def** The *degree* of a vertex i is $\deg(i) = \#\text{edges connected to } i$

$$\deg(i) = S_{N-1} \sim \mathrm{Binom}(N-1, p) \Rightarrow \mathbb{E}\deg(i) = \boxed{(N-1)p =: d}$$

"Expected degree"

- Phase transitions $\quad d \simeq 1$: giant component; $\quad d = \log n$: connectivity $\Big\}$ (see Wikipedia)

Def a graph is d-regular if $\deg(i) = d \ \forall i$



2-regular 3-regular

We will show: $d \sim \log n$ is a phase transition for regularity of $G(n,p)$: $\begin{cases} d \gg \log n \Rightarrow \text{almost } d\text{-regular} \\ d \ll \log n \Rightarrow \text{very far from it} \end{cases}$

THM (Regularity of dense random graphs)

$\exists$ abs. const $C>0$ such that if $d \geq C \log n$
then $G(n,p)$ is almost d-regular with high prob:

$$\mathbb{P}\{\forall i: \underbrace{0.9d \leq \deg(i) \leq 1.1d}_{E_i}\} \geq 0.9.$$

Proof $\forall i$, Chernoff(p.1) $\Rightarrow \mathbb{P}(E_i^c) = \mathbb{P}\{|\deg(i) - d| \geq 0.1d\}$

Binomial with mean μ

$$\leq \exp\left(-\frac{0.1^2 d}{6}\right) \leq \exp\left(-\frac{0.1^2 C \log N}{6}\right)$$

$\leq \frac{1}{10N}$ if we choose C a large const.

Union Bd:

$$\mathbb{P}\left(\bigcup_{i=1}^N E_i^c\right) \leq \sum_{i=1}^N \mathbb{P}(E_i^c) \leq N \cdot \frac{1}{10N} = \frac{1}{10}.$$

$\Rightarrow \mathbb{P}\left(\bigcap_{i=1}^N E_i\right) \geq 1 - \frac{1}{10} = \frac{9}{10}$ (de Morgan law). QED