HOMEWORK 8

HDP KNU+ FALL 2022

Hints are in the back of this homework set.

We showed in Lecture 22 (October 21) that the operator and Frobenius norms are equivalent up to a factor of $\sqrt{n}$: the two-sided bound

$$\|A\| \leq \|A\|_F \leq \sqrt{n}\|A\|_F$$

holds for any $n \times n$ matrix A. In the first problem, you will check that this result is optimal. there are matrices for which the two norms are equal, and there are matrices for which the two norms are $\sqrt{n}$ apart.

PROBLEM 1 (THE GAP BETWEEN OPERATOR AND FROBENIUS NORMS)

(a) Compute the operator and Frobenius norms of the identity matrix.

(b) Compute the operator and Frobenius norms of the matrix whose all entries = 1.

In this problem, we express the operator norm as the maximum of a quadratic form. This interpretation will be crucial in our analysis of covariance estimation.

PROBLEM 2 (OPERATOR NORM AND QUADRATIC FORMS)

Let A be an $n \times n$ symmetric matrix.

(a) Show that

$$\|A\| = \max_{i=1,\ldots,n} |\lambda_i(A)|\,,$$

where $\lambda_i(A)$ denote the eigenvalues of A.

(b) Show that

$$\|A\| = \max_{x \in S^{n-1}} \left|x^{\mathsf{T}} A x\right|,$$

where S^{n-1} denotes the unit Euclidean sphere, i.e. the set of unit vectors in $\mathbb{R}^n$.

(c) Show by example that the formula in (b) may fail for non-symmetric matrices.

You may recall that the normal distribution is stable: any linear combination of independent normal random variable is again normal. More is true: some linear combinations of independent normals can again be *independent* normals.

PROBLEM 3 (COMBINATIONS OF GAUSSIANS)

Let X_1, X_2 be independent $N(0, 1)$ random variables. Show that $Y_1 = (X_1 + X_2)/\sqrt{2}$ and $Y_2 = (X_1 - X_2)/\sqrt{2}$ are independent $N(0, 1)$ random variables.

In Problem 2(b) we showed how to compute the operator norm of A by maximizing the quadratic form $x^{\mathsf{T}}Ax$ over the unit sphere. Now we will run a discretization argument: we replace the sphere with its ε-net. This obviously makes the maximum smaller. However, we show that the drop is not that big: we will show that the net still captures the operator norm quite accurately. This discretization will be crucial in our analysis of covariance estimation.

Problem 4 (Computing the operator norm on a net)

Let A be an $n \times n$ symmetric matrix, $\varepsilon \in (0, 1/2)$, and $\mathcal{N}$ be an ε-net of the unit sphere S^{n-1}. Show that

$$\|A\| \leq \frac{1}{1-2\varepsilon} \cdot \max_{x \in \mathcal{N}} \left| x^{\mathsf{T}}Ax \right| .$$

TURN OVER FOR HINTS

Hints

Hint for Problem 2.

(a) The operator norm of A is the maximal singular value of A. Since the matrix A is symmetric, its singular values equal $|\lambda_i(A)|$ (why?)

(b) Use the spectral decomposition to reduce the problem to a diagonal matrix, then solve it for a diagonal matrix directly.

(c) If A is a rotation of the plane by $90°$, the vectors x and Ax are always orthogonal.

Hint for Problem 3. Find an orthogonal map that transforms the vector (X_1, X_2) into (Y_1, Y_2), and use rotation invariance (Lecture 22, October 2).

Hint for Problem 4. The overall approach can be similar to the proof of a similar result for non-symmetric matrices (Lecture 23, October 24, or the proof of Lemma 4.4.1 in the book). Choose x that maximizes the quadratic form on the sphere, and approximate it by some u in the net. To bound the norm of the difference between $x^\mathsf{T} Ax$ and $u^\mathsf{T} Au$, bound the norm of the difference between each of these two quantities to $x^\mathsf{T} Au$ and use triangle inequality.