# LECTURE 14

In lecture 13, we proved Goemans-Willamson's Max Cut Theorem: $\forall (a_{ij})_{i,j=1}^n$:

NP-hard ☹ SDP ☺

$$\max_{x_i=\pm1} \sum_{i,j=1}^n a_{ij}(x_i-x_j)^2 \geq 0.878 \cdot \max_{\substack{v_i\in\mathbb{R}^d \\ \text{unit vectors}}} \sum_{i,j=1}^n a_{ij}\|v_i-v_j\|_2^2 \qquad (*)$$

Remark: a version with $(x_i-y_j)^2$ in L.H.S and $\|u_i-v_j\|_2^2$ in R.H.S. is also valid and follows from proof

Today, we prove:

THM (Grothendieck's inequality '1953) $\forall (a_{ij})$, $\forall d$:

$$\max_{\substack{u_i,v_j\in\mathbb{R}^d \\ \text{unit vectors}}} \sum_{ij} a_{ij}\langle u_i,v_j\rangle \leq 1.782 \cdot \max_{x_i,y_j\in\{\pm1\}} \sum_{ij} a_{ij}x_iy_j$$

SDP ☺ NP-hard ☹

Proof again is based on Grothendieck's identity (previous lecture + HW):

$$\mathbb{E}\,\text{sign}\langle u,g\rangle\,\text{sign}\langle v,g\rangle = \frac{2}{\pi}\arcsin\langle u,v\rangle \qquad \forall \text{ unit vectors } u,v\in\mathbb{R}^d$$

nonlinearity ☹ What if there were NO arcsin?

- WRONG PROOF assuming $\mathbb{E}\,\text{sign}\langle u,g\rangle\,\text{sign}\langle v,g\rangle = \langle u,v\rangle$:

$$\sum_{ij} a_{ij}\langle u_i,v_j\rangle = \sum_{i,j} a_{ij}\,\mathbb{E}\,\text{sign}\langle u_i,g\rangle\,\text{sign}\langle v,g\rangle = \mathbb{E}\sum_{ij} a_{ij}\,\underbrace{\text{sign}\langle u_i,g\rangle}_{\in\{\pm1\}}\,\underbrace{\text{sign}\langle v_j,g\rangle}_{\in\{\pm1\}}$$

$$\leq \max_{x_i,y_j\in\{\pm1\}} \sum_{ij} a_{ij}x_iy_j \qquad \square.$$

- How can we handle the nonlinearity? linearize as in proof of $(*)$?

Won't work here.

- **KERNEL TRICK**: We will find maps $\phi, \psi : S^{d-1} \to S^{N-1}$ ← i.e. unit vectors $\downarrow \phi,\psi$ unit vectors

$$\frac{2}{\pi} \arcsin \langle \phi(u), \psi(v) \rangle = \beta \langle u, v \rangle \quad \forall u, v \in S^{d-1}$$

where $\beta \simeq 0.561$.

- **CORRECT PROOF** assuming the Kernel Trick [Krivine's argument 1979]

$$\sum_{i,j} a_{ij} \langle u_i, v_j \rangle \overset{\text{Kernel Trick}}{=} \frac{1}{\beta} \sum_{i,j} a_{ij} \cdot \frac{2}{\pi} \arcsin \langle \underbrace{\phi(u_i)}_{= u_i'}, \underbrace{\psi(v_j)}_{= v_j'} \rangle \overset{\text{Grothendieck identity}}{=} \frac{1}{\beta} \sum_{i,j} a_{ij} \, \mathbb{E} \operatorname{sign} \langle u_i', g \rangle \operatorname{sign} \langle v_j', g \rangle$$

$$= \frac{1}{\beta} \mathbb{E} \sum_{i,j} a_{ij} \underbrace{\operatorname{sign} \langle u_i, g \rangle}_{\in \{\pm 1\}} \underbrace{\operatorname{sign} \langle v_j, g \rangle}_{\in \{\pm 1\}}$$

$$\le \underbrace{\frac{1}{\beta}}_{\approx 1.782} \max_{x_i, y_j \in \{\pm 1\}} \sum_{i,j} a_{ij} x_i y_j \qquad \square$$

- Kernel trick $\iff$ $\boxed{\langle \phi(u), \psi(v) \rangle = \sin\left( \frac{\beta \pi}{2} \langle u, v \rangle \right)}$ How?

- PLAN: We will build more and more nontrivial nonlinearities:

$$\langle u, v \rangle^2 \longrightarrow \langle u, v \rangle^k \longrightarrow \forall \text{ polynomial} \longrightarrow \forall \text{ analytic function}$$

e.g. $\sin(\cdot)$

# TENSOR CALCUCUS

① For a vector $u \in \mathbb{R}^n$, <u>tensor product</u>

$$u \otimes u := [u_i u_j]_{i,j=1}^{n} = u u^T \in \mathbb{R}^{n^2}$$

- $\langle u \otimes u, v \otimes v \rangle = \sum_{i,j} u_i u_j \cdot v_i v_j = \left(\sum u_i v_i\right)\left(\sum u_j v_j\right) = \langle u, v \rangle^2$

- $\Rightarrow$ $\exists$ map $\phi : \mathbb{R}^n \to \mathbb{R}^{n^2}$ such that

$$\langle \phi(u), \phi(v) \rangle = \langle u, v \rangle^2 \qquad (\phi(u) = u \otimes u)$$

unit vectors $\mapsto$ unit vectors.

SQUARE Built.

② More generally,

$$u \otimes u \otimes u = [u_i u_j u_k]_{i,j,k=1}^{n} \in \mathbb{R}^{n^3}$$

and $\qquad u^{\otimes k} \in \mathbb{R}^{n^k}$

$\Rightarrow$ $\exists$ map $\phi : \mathbb{R}^n \to \mathbb{R}^{n^2}$ :

$$\langle \phi(u), \phi(v) \rangle = \langle u, v \rangle^k \qquad (\phi(u) = u^{\otimes k})$$

unit vectors $\to$ unit vectors.

$\forall$ MONOMIAL Built.

③ Direct sum of $u \in \mathbb{R}^n$, $v \in \mathbb{R}^m$:

$$u \oplus v = [u_1, \dots, u_n, v_1, \dots, v_m] \in \mathbb{R}^{n+m}.$$

$$u \oplus v \oplus w = \text{etc.}$$

$$\langle u \oplus v, x \oplus y \rangle = \langle u, x \rangle + \langle v, y \rangle. \qquad (\cancel{*}*)$$

• Ex $\exists \ \phi : \mathbb{R}^n \to \mathbb{R}^N$ : $\langle \phi(u), \phi(v) \rangle = 2\langle u, v \rangle + 3\langle u, v \rangle^2 + 5\langle u, v \rangle^3.$

$\ulcorner$ $\phi(u) = (\sqrt{2}\, u) \oplus (\sqrt{3}\, u \otimes u) \oplus (\sqrt{5}\, u \otimes u \otimes u) \in \mathbb{R}^{n+n^2+n^3}$ $\lrcorner$

$\Rightarrow \forall$ polynomial with nonnegative coeffs built.

④ If some coeff's are negative, impossible $(\langle \phi(u), \phi(u) \rangle \geq 0)$

But:

• Ex $\exists \ \phi, \psi : \mathbb{R}^n \to \mathbb{R}^N$ : $\langle \phi(u), \psi(v) \rangle = 2\langle u, v \rangle - 3\langle u, v \rangle^2 - 5\langle u, v \rangle^3.$

$\ulcorner$ $\phi(u)$ as above;

$\psi(u) := (\sqrt{2}\, u) \oplus (-\sqrt{3}\, u \otimes u) \oplus (-\sqrt{5}\, u \otimes u \otimes u).$ $\lrcorner$

$\Rightarrow \forall$ polynomial built

Remark: $\forall$ unit $u$: $\quad \|\phi(u)\|_2^2 = \|\psi(u)\|_2^2 = 2+3+5 = 10.$

(5) Take limits $\Rightarrow$ $\forall$ real analytic function Built:

**Prop** $\forall$ real analytic function $f(x) = \sum_k c_k x^k$, $\forall \varepsilon > 0$

$\exists\ \phi, \psi : \mathbb{R}^n \to \ell^2$ (define it!)

$$\langle \phi(u), \psi(v) \rangle = f(\langle u, v \rangle) \quad \forall u, v \in \mathbb{R}^n.$$

Moreover,
$$\|\phi(u)\|_2^2 = \|\psi(u)\|_2^2 = \sum_k |c_k|$$

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Now we are ready to prove ($*$ p.1):

**Lem** $\exists\ \phi, \psi : \mathbb{R}^n \to \ell^2$ that take unit vectors to unit vectors,

and
$$\langle \phi(u), \psi(v) \rangle = \sin\Big( \underbrace{\tfrac{\beta\pi}{2}}_{= c} \langle u, v \rangle \Big) \quad \forall\ u, v \text{ unit}$$

where $\beta = \frac{2}{\pi} \ln(1 + \sqrt{2})$.

Apply Prop. for

$$f(x) = \sin(cx) = cx - \frac{(cx)^3}{3!} + \frac{(cx)^5}{5!} - \frac{(cx)^7}{7!} + \dots$$

$\phi$ maps unit vectors to unit vectors if

$$1 = \sum_k |c_k| = c + \frac{c^3}{3!} + \frac{c^5}{5!} + \frac{c^7}{7!} + \dots = \frac{e^c - e^{-c}}{2}$$

Solve $\Rightarrow$ $c = \ln(1 + \sqrt{2}) =: \frac{\beta\pi}{2}$ $\Rightarrow$ $\beta = \frac{2}{\pi} \ln(1 + \sqrt{2})$.

**Q:** Is const $1.782\ldots$ optimal? No [Braverman et al. 2011]