LECTURE 17

- Refresher in Linear Algebra contd.

Last class:

SEE MORE IN KDP-2022

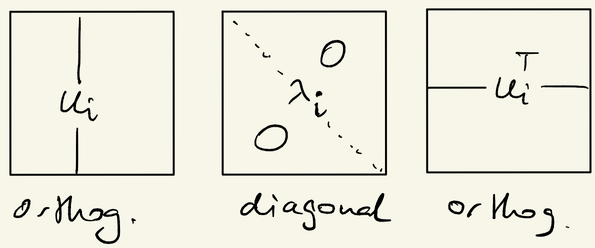
Thm (Spectral decomposition) $\forall$ symmetric $n\times n$ matrix A:

$$A = \sum_{i=1}^{r} \lambda_i u_i u_i^T \quad \text{where } r = \operatorname{rank}(A)$$

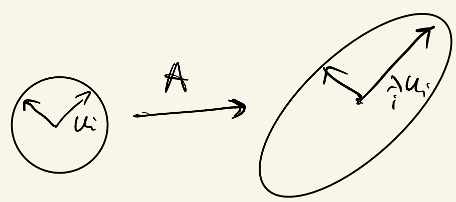
and (λ_i, u_i) are eigenvalues & unit eigenvectors of A.

(a) Matrix form: $A = U \Lambda U^T$

$$A = U \cdot \Lambda \cdot U^T$$



(b) Geometric form



(c) Optimization form:

$$\lambda_1 = u_1^T \overbrace{A u_1}^{= \lambda u_1} = \max_{x \text{ unit}} x^T A x, \quad u_1 = \operatorname{argmax} \ldots \quad \lambda_i \text{ times}$$

$$\lambda_2 = u_2^T A u_2 = \max_{\substack{x \perp u_1 \\ \text{unit}}} x^T A x, \quad u_2 = \operatorname{argmax} \ldots$$

$$\lambda_3 = u_3^T A u_3 = \max_{\substack{x \perp \{u_1, u_2\} \\ \text{unit}}} x^T A x, \quad u_3 = \operatorname{argmax} \ldots$$

$$\cdots$$

$$\lambda_n = u_n^T A u_n = \min_{x \text{ unit}} x^T A x, \quad u_n = \operatorname{argmin} \ldots$$

- For general matrices, non-symmetric, rectangular:

THM (Singular value decomposition – SVD)

$\forall$ $m \times n$ matrix A can be expressed as

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T \qquad (*)$$

where $r = \text{rank}(A)$ and:

$\sigma_1 \geq \sigma_2 \geq \dots > 0$ "singular values";

$u_1, \dots, u_m \in \mathbb{R}^m$ are orthonormal "left-singular vectors"

$v_1, \dots, v_n \in \mathbb{R}^n$ are orthonormal "right-singular vectors"

- Relation between singular values/vectors $\longleftrightarrow$ eigenvalues/vectors ?

$$AA^T = \left(\sum_i \sigma_i u_i v_i^T\right)\left(\sum_j \sigma_j v_j u_j^T\right) = \sum_{ij} \sigma_i \sigma_j \, u_i \underbrace{v_i^T v_j}_{\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \end{cases}} u_j^T = \sum_i \sigma_i^2 u_i u_i^T$$

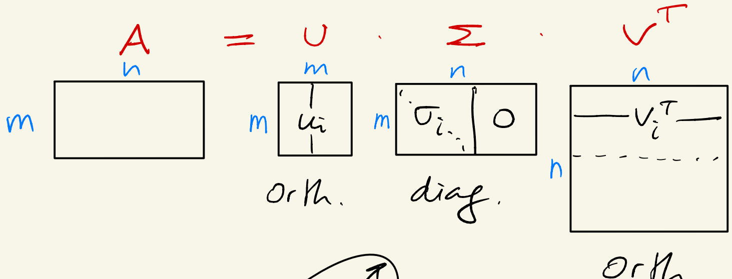
Similarly, $A^T A = \sum_i \sigma_i^2 v_i v_i^T$

$$\sigma_i(A) = \sqrt{\lambda_i(AA^T)} = \sqrt{\lambda_i(A^T A)}.$$

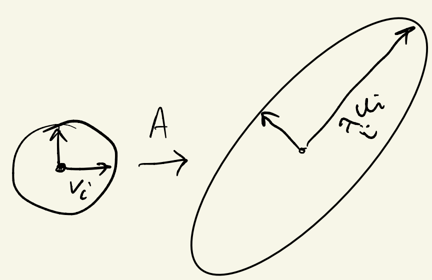
$u_i(A)$ = eigenvectors of AA^T; $v_i(A)$ = eigenvectors of $A^T A$

(a) Matrix form:

$$A = U \Sigma U^T$$



(b) Geometric form:



(c) Optimization form:

$$\sigma_1 = \max_{x,y \text{ unit}} x^T A y = \max_{x \text{ unit}} \|Ax\|_2, \qquad v_1 = \text{argmax} \dots$$

$$\sigma_2 = \max_{\substack{x \perp v_1 \\ \text{unit}}} \|Ax\|_2, \qquad v_2 = \text{argmax} \dots$$

$$\sigma_3 = \max_{\substack{x \perp \{v_1, v_2\} \\ \text{unit}}} \|Ax\|_2, \qquad v_3 = \text{argmax} \dots$$

$\dots \qquad \dots$

$$\sigma_n = \min_{x \text{ unit}} \|Ax\|_2 \qquad v_n = \text{argmin} \dots$$

More general optimization form:

Min-Max Theorem $\forall$ symmetric $n\times n$ A, $\forall k$:

$$\lambda_k(A) = \max_{\dim E=k} \min_{\substack{x\in E\\ \text{unit}}} x^T A x = \min_{\dim F = n-k+1} \max_{\substack{x\in F\\ \text{unit}}} x^T A x$$

Moreover, the optimal E, F are spanned by eigenvectors of A:
$E = \text{span}\{u_1, \dots, u_k\}$, $F = \text{span}\{u_k, \dots, u_n\}$

The optimal x is the eigenvector u_k

- Apply MMT for $A^T A$ $\Rightarrow$ get a MMT for singular values/vectors:

$$\sigma_k(A) = \max_{\dim E=k} \min_{\substack{x\in E\\ \text{unit}}} \|Ax\|_2 = \min_{\dim F=n-k+1} \max_{\substack{x\in F\\ \text{unit}}} \|Ax\|_2$$

MATRIX NORMS

A: $m\times n$ matrix

Def Frobenius (a.k.a. Hilbert-Schmidt) norm:

$$\|A\|_F^2 = \sum_{i,j} A_{ij}^2$$

= Euclidean norm on $\mathbb{R}^{m\times n}$.

$$\langle A, B\rangle = \sum_{i,j} A_{ij} B_{ij} = \text{tr}(A^T B). \qquad (*)$$

$$\Rightarrow \langle A, A\rangle = \|A\|_F^2$$

Prop (Orthogonal invariance) $\forall$ orthogonal U, V:

$$\|UA\|_F = \|AV\|_F = \|A\|_F$$

$$\Big[\ \|UA\|_F = \langle UA, UA\rangle \overset{(*)}{=} \text{tr}(A^T \underbrace{U^T U}_{=I} A) = \text{tr}(A^T A) = \langle A, A\rangle = \|A\|_F^2\ \Big]$$

Prop $\|A\|_F^2 = \sum_i \sigma_i(A)^2$ (spectral!)

singular values

$$\Big[\ \|A\|_F^2 \overset{\text{SVD}}{=} \|U^T \Sigma V\|_F^2 = \|\Sigma\|_F^2 = \sum \sigma_i(A)^2\ \Big]$$

orthog. invariance   diag(σ_i)