# LECTURE 21

- Work with matrices as with numbers?

  random matrices $\sim$ random variables?

  $\leftarrow$ CLT, LLN, concentration

## FUNCTIONAL CALCULUS

Symmetric $d \times d$ matrices.

- Adding, subtracting, multiplying, dividing $(A/B = AB^{-1})$
- $0$; $1 \mapsto I$
- $|a| \to \|A\|$ (operator norm)
- Functions of matrices: if $A = \sum_{i=1}^{d} \lambda_i u_i u_i^T$ (spectral decomposition)

then $A^2 = \sum_{i=1}^{d} \lambda_i^2 u_i u_i^T$, $\quad A^{-1} = \sum_{i=1}^{d} \frac{1}{\lambda_i} u_i u_i^T$,

$$I + A + A^2 = \sum_{i=1}^{d} (1 + \lambda_i + \lambda_i^2) u_i u_i^T$$

Similarly, $\forall$ polynomial: $p(A) = \sum_{i=1}^{d} p(\lambda_i) u_i u_i^T \quad \Rightarrow$

**Def** (Function of a matrix)

If $f: \mathbb{R} \to \mathbb{R}$ and $A = \sum_{i=1}^{d} \lambda_i u_i u_i^T \Rightarrow f(A) := \sum_{i=1}^{d} f(\lambda_i) u_i u_i^T$

**Ex**

$$e^A = \sum_{i=1}^{d} e^{\lambda_i} u_i u_i^T = \sum_{i=1}^{d} 1 \cdot u_i u_i^T + \sum_{i=1}^{d} \lambda_i u_i u_i^T + \frac{1}{2!} \sum_{i=1}^{d} \lambda_i^2 u_i u_i^T + \dots$$

$$e^{\lambda_i} = 1 + \lambda_i + \frac{1}{2!}\lambda_i^2 + \dots$$

$$= I + A + \frac{1}{2!} A^2 + \dots$$

(Matrix Taylor series)

# LÖWNER ORDER

on the set of $d{\times}d$ symmetric matrices.

Recall: $A$ is PSD $\overset{\text{def}}{\iff} u^T A u \geq 0 \;\forall u \in \mathbb{R}^d \iff \lambda_i(A) \geq 0 \;\forall i$.

**Def (Löwner order)** We say that $A \succcurlyeq 0$ if $A$ is PSD.
We say $A \succcurlyeq B$ and $B \preccurlyeq A$ if $A - B \succcurlyeq 0$

- $\preccurlyeq$ is a partial order, but not a total order: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is neither $\succcurlyeq 0$ nor $\preccurlyeq 0$.

  **HW**

- **WARNING**: matrix multiplication is non-commutative:

$$XY \neq YX$$

$\Rightarrow$ some basic facts don't transfer from scalars to matrices.

e.g. $X \succcurlyeq 0,\; Y \succcurlyeq 0 \not\Rightarrow \underbrace{XY}_{} \succcurlyeq 0$

needs not be symmetric!

**HW**

**Prop (Eigenvalue monotonicity)** $X \succcurlyeq Y \Rightarrow \lambda_i(X) \leq \lambda_i(Y) \;\forall i$

**Proof**: Courant-Fisher min-max theorem:

$$\lambda_i(X) = \max_{\dim E = i} \; \min_{\substack{u \in E \\ \text{unit}}} \; \underset{\substack{\wedge \\ u^T Y u}}{u^T X u} \qquad \square.$$

- **Remark** The converse holds for commuting matrices but not in general

**HW**

**Cor (Trace monotonicity)**
Let $f: \mathbb{R} \to \mathbb{R}$ be an increasing function. Then

$$X \preccurlyeq Y \Rightarrow \operatorname{tr} f(X) \leq \operatorname{tr} f(Y)$$

$$\left[ \sum_{i=1}^{d} f(\lambda_i(X)) \leq \sum_{i=1}^{d} f(\lambda_i(Y)) \text{ by Prop. \& increasing } f \right]$$

- Inequalities in 1 variable generalize to matrices: $f(x) \le g(x)\ \forall x \in \mathbb{R} \Rightarrow f(X) \preceq g(X)$ $\forall$ Symmetric $X$ (by def)

  but not in 2 variables!

  "$X \preceq Y \implies f(X) \preceq f(Y)$"

  holds for some monotone functions like $\frac{1}{x}$ and $\ln(x)$ **HW**

  but fails for others like $x^2$.

- General theory of "matrix monotone functions": [K. Löwner '1936]

- Matrix versions of Hoeffding, Chernoff, Bernstein?

  Yes: [Tropp '2010]

  BIG ISSUE: $e^{A+B} \neq e^A e^B$ in general **HW**