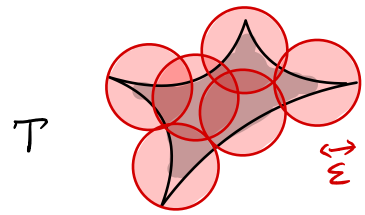
LECTURE 3

- Application of Approximate Caratheodory Theorem for:

COVERING NUMBERS

Def The covering number of a set $T \subset \mathbb{R}^n$ at scale $\varepsilon > 0$ is the smallest number of Euclidean balls of radius ε needed to cover T. Denoted $N(T, \varepsilon)$.

Ex



$N(T, \varepsilon) \leq 6$

- Covering numbers is a measure of complexity of T (like area, volume ...)
- Suffer from the curse of high dimensionality:

Prop For $B =$ unit Euclidean ball in $\mathbb{R}^n$,

$$N(B, \varepsilon) \geq \left(\frac{1}{\varepsilon}\right)^n \quad \forall \varepsilon > 0$$

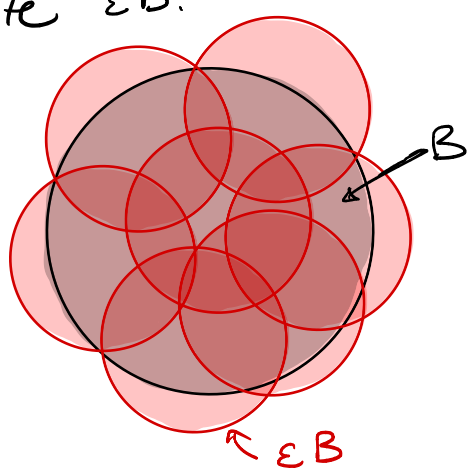
Exponential in dimension!

Proof Assume B can be covered by N copies of a ball of radius ε, which we denote εB.

Comparing the volumes gives

$$\mathrm{Vol}(B) \leq N \cdot \underbrace{\mathrm{Vol}(\varepsilon B)}_{= \varepsilon^n \cdot \mathrm{Vol}(B)}$$

$$\Rightarrow N \geq \frac{1}{\varepsilon^n}. \qquad \text{QED.}$$



- Generally, covering #s are exponential in the dimension, (e.g. for cube) But not always:

Polytopes with few vertices have small covering numbers:

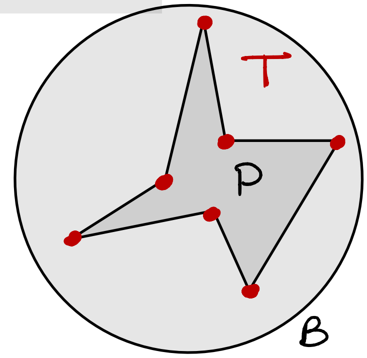
↑ багатогранники"

THM Let P be a polytope in $\mathbb{R}^n$ with m vertices, $P \subset B$ (↑ unit Euclidean Ball).
Then $N(P, \varepsilon) \le m^{1/\varepsilon^2}$ $\quad \forall \varepsilon > 0$.

Dimension-free. ↑Polynomial in #vertices

Proof is based on a version of Approximate Caratheodory Thm (HW2):

(*) $\forall$ set $T \subset B$, $\forall x \in \mathrm{conv}(T)$ $\quad \forall k \in \mathbb{N}$ $\quad \exists\, x_1, \dots, x_k \in T$:
$$\Big\| x - \frac{1}{k} \sum_{i=1}^{k} x_i \Big\|_2 \le \frac{1}{\sqrt{k}}.$$



Proof of THM:

- $P \subset \mathrm{conv}(T)$ where $T = \{\text{vertices of } P\} \subset B$
- A.C.T (*) $\Rightarrow$

  $\forall x \in P \subset \mathrm{conv}(T)$ is within distance $\frac{1}{\sqrt{k}}$ from some point in the set
  $$\mathcal{N} := \Big\{ \frac{1}{k} \sum_{i=1}^{k} x_i \; : \; x_i \in T \Big\}$$

$\Rightarrow$ $\forall x \in P$ is covered by a ball of radius $\frac{1}{\sqrt{k}}$ and center $\in \mathcal{N}$.

$\Rightarrow$ $N\big(P, \frac{1}{\sqrt{k}}\big) \le |\mathcal{N}| \le m^k$

↑ # ways to choose k elements x_i from the set T of m elements, with repetition

- Choose k: $\frac{1}{\sqrt{k}} = \varepsilon$ $\quad \left(k = \frac{1}{\varepsilon^2} \right) \Rightarrow$ QED

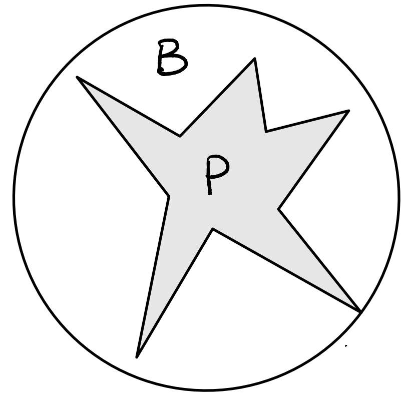
Application: polytopes with few vertices have exponentially small volume:

THM [Carl–Pajor '88]

Let $P \subset B$ polytope with m vertices

B: unit Euclidean Ball.

Then
$$\frac{\mathrm{Vol}(P)}{\mathrm{Vol}(B)} \le \Big(\underbrace{3\sqrt{\frac{\log m}{n}}}_{\le 2^{-n} \text{ if } m < \exp(cn)}\Big)^n.$$



Proof By def. of covering numbers, P can be covered by $N(P,\varepsilon)$ copies of a ball of radius ε, denoted εB.

Comparing the volumes yields
$$\mathrm{Vol}(P) \le N(P,\varepsilon) \cdot \mathrm{Vol}(\varepsilon B).$$

- By (*) p.3, we have $N(P,\varepsilon) \le m^{1/\varepsilon^2}$
- By volume scaling, $\mathrm{Vol}(\varepsilon B) = \varepsilon^n \cdot \mathrm{Vol}(B)$.

Substitute $\Rightarrow$
$$\frac{\mathrm{Vol}(P)}{\mathrm{Vol}(B)} \le m^{1/\varepsilon^2} \cdot \varepsilon^n \qquad \forall \varepsilon > 0.$$

- Minimize RHS in ε $\Rightarrow$ QED. (DIY: take logs & differentiate $\Rightarrow$ optimal $\varepsilon = \sqrt{\frac{2\log m}{n}}$)

Remarks

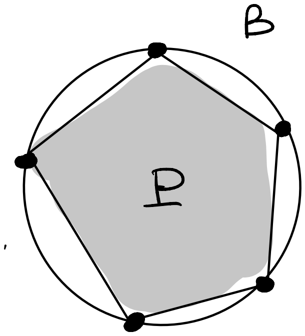
① [Carl-Pajor] proved a slightly sharper bound $\left(C\sqrt{\frac{\log(m/n)}{n}}\right)^n$.

② It is optimal, attained by a random polytope

$$P = \mathrm{conv}\{x_1, \dots, x_m\}$$

where x_i = indep. random pts on the unit sphere.

[Dafnis-Giannopoulos-Tsolomitis '2003, 2009]



hyperbolic intuition

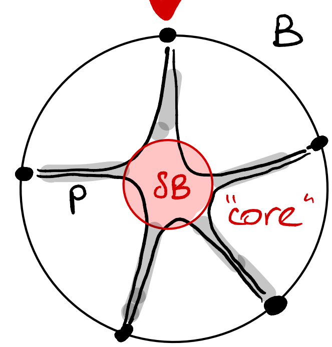
③ "Hyperbolic intuition"

for $\delta := \sqrt{\frac{4\log m}{n}}$, Thm says:

tiny ball!

$$\mathrm{Vol}(P) \le \delta^n \, \mathrm{Vol}(B) = \mathrm{Vol}(\delta B).$$

Thus the picture "actually" looks like this, even though it violates convexity.



V. Milman's "hyperbolic intuition"

④ A "blind spider" experiment:

- A random walk will stay in the "core" δB, will not find the "tentacles".

- The spider can't tell the polytope from the ball δB.

- Bad for algorithms (MCMC)