LECTURE 9

SPACES OF RANDOM VARIABLES

"species" → "habitat"
- numbers → fields (Galois)
- transformations → groups
- polynomials → rings
- r.v.'s → function spaces

① Normed spaces.

Def A <u>norm</u> on a linear vector space V is a function $\|\cdot\| : M \to \mathbb{R}$ s.t.

1. $\|x\| \geq 0 \quad \forall x \in V$
2. $\|x\| = 0$ implies $x = 0$
3. $\|\alpha x\| = |\alpha| \, \|x\| \qquad \forall x \in V, \ \alpha \in \mathbb{R}$
4. $\|x + y\| \leq \|x\| + \|y\|$

The pair $(V, \|\cdot\|)$ is called a <u>normed space</u>

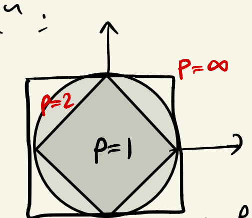
Ex (a) The <u>Euclidean norm</u> on $\mathbb{R}^n$:

$$\|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$$

$(\mathbb{R}^n, \|\cdot\|_2)$ is a normed sp.

(b) More generally, for $p \in [1, \infty]$, the <u>ℓ^p norm</u> on $\mathbb{R}^n$:

$$\|x\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}, \qquad \|x\|_\infty = \max_i |x_i|$$



unit balls of ℓ^p

<u>NOT</u> a norm $\forall p < 1$.

(c) For $p \in [1, \infty]$, the <u>L^p norm</u> of a random variable X

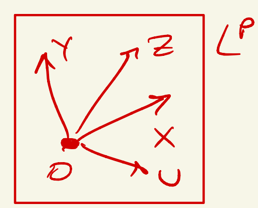
$$\|X\|_{L^p} := \left(\underbrace{\mathbb{E}|X|^p}\right)^{1/p}, \qquad \|X\|_{L^\infty} = \text{ess.sup}\,|X|$$

← "p-th absolute <u>moment</u> of X"

$L^p := \{\text{all r.v.'s } X \text{ such that } \|X\|_{L^p} < \infty\}$.

↑ on the same prob. space

r.v.'s = vectors;



e.g: $L^1 = \{$r.v.'s with finite expectation$\}$
$L^2 = \{$r.v.'s with finite expectation & variance$\}$
$L^\infty = \{$r.v.'s that are bounded a.s.$\}$

Remark (a link to the <u>functional space</u> L^p)

r.v. = <u>function</u> on Ω (prob. space),
expectation = Lebesgue integral $\quad \mathbb{E}X = \int_\Omega X(\omega)\, dP(\omega)$

An important class of normed spaces is:

② Hilbert spaces

Def A (real) inner product on a vector space H is a function $\langle \cdot , \cdot \rangle : H \times H \to \mathbb{R}$ s.t.

1. $\langle x, x \rangle \geq 0 \quad \forall x \in H$
2. $\langle x, x \rangle = 0$ implies $x \in H$.
3. $\langle x, y \rangle = \langle y, x \rangle \quad \forall x, y \in H$
4. $\langle ax + bz, y \rangle = a \langle x, y \rangle + b \langle z, y \rangle \quad \forall x, y, z \in H, \ \forall a, b \in \mathbb{R}$.

Fact $\forall$ Hilbert space is a normed space if we set
$$\|x\| := \sqrt{\langle x, x \rangle}.$$

Ex (a) $\mathbb{R}^n$ with the Euclidean inner product
$$\langle x, y \rangle := \sum_{i=1}^n x_i y_i \quad \Rightarrow \quad \sqrt{\langle x, x \rangle} = \|x\|_2.$$

(b) L^2 with the inner product
$$\langle X, Y \rangle := \mathbb{E}[XY] \quad \Rightarrow \quad \sqrt{\langle X, X \rangle} = \left(\mathbb{E}[X^2]\right)^{1/2} = \|X\|_{L^2}$$

③ Euclidean geometry of random variables:

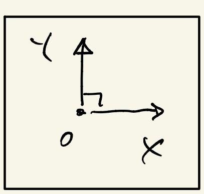
Consider the linear subspace
$$E := \{ \text{r.v.'s } X \text{ with } \mathbb{E}X = 0 \} \subset L^2.$$

On this subspace

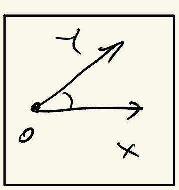
$$\langle X, Y \rangle = \mathbb{E}[XY] = \underset{\text{covariance}}{\mathrm{Cov}(X, Y)}$$

$$\|X\|^2 = \mathbb{E}[X^2] = \mathrm{Var}(X)$$

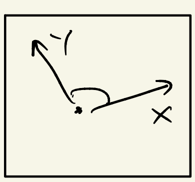
$$\|X\| = \sqrt{\mathrm{Var}(X)} = \underset{\text{standard dev.}}{\sigma(X)}$$



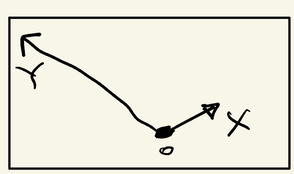
uncorrelated



positively correlated



negatively correlated



X has smaller variance than Y

INEQUALITIES

① Cauchy – Schwarz : $\forall$ Hilbert space $\mathcal{H}$,

$$|\langle x,y\rangle| \le \|x\|\cdot\|y\|$$

For L^2, this reads:

$$|\mathbb{E}(XY)| \le \|X\|_{L^2}\,\|Y\|_{L^2} = (\mathbb{E}X^2)^{1/2}(\mathbb{E}Y^2)^{1/2} \qquad (*)$$

apply for $X \mapsto X-\mathbb{E}X$, $\quad Y \to Y-\mathbb{E}Y$

$$|Cov(X,Y)| \le \sqrt{Var(X)}\cdot\sqrt{Var(Y)}.$$

② Hölder : more general than $(*)$:

$$|\mathbb{E}(XY)| \le \|X\|_{L^p}\,\|X\|_{L^q} \qquad \forall p,q>0:\ \frac{1}{p}+\frac{1}{q}=1.$$

$p=q=2 \Rightarrow (*)$

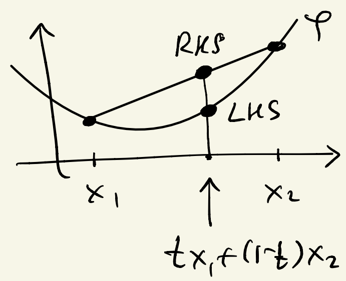
③ Jensen:

THM (Jensen's ineq.): Let X be a r.v. and $\varphi:\mathbb{R}\to\mathbb{R}$ be a convex function. Then

$$\varphi(\mathbb{E}[X]) \le \mathbb{E}[\varphi(X)]$$

i.e.

$$(tx_1+(1-t)x_2) \le t\varphi(x_1)+(1-t)\varphi(x_2)$$



Ex Two-valued r.v:

$P\{X=a\}=p,\quad P\{X=b\}=1-p$

J.I:

$$\varphi(pa+(1-p)b) \le p\varphi(a)+(1-p)\varphi(b)$$

$\Leftrightarrow$ def. of convexity.

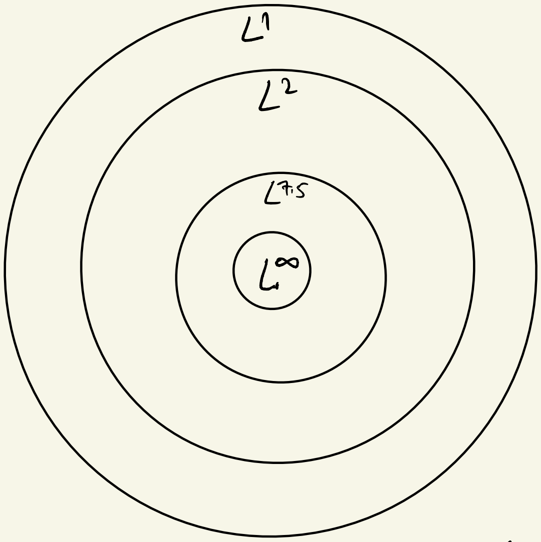
Cor: $\|X\|_{L^p}$ increases in p

Γ If $p \le q$, $\varphi(x) = x^{q/p}$ is convex $\Rightarrow$ J.I for $|X|^p$

$$\varphi(\mathbb{E}|X|^p) \le \mathbb{E}\varphi(|X|^p)$$

$$\left(\mathbb{E}|X|^p\right)^{q/p} \le \mathbb{E}|X|^q \xrightarrow{\text{take } q^{th} \text{ root}} \|X\|_{L^p} \le \|X\|_{L^q}. \ \lrcorner$$

- $\lim_{p\to\infty} \|X\|_{L^p} = \|X\|_{L^\infty} = \text{ess.sup}|X|.$



$\Rightarrow L^1 \supset L^2 \supset L^{7.5} \supset \cdots \supset L^\infty$

- QUESTION: Is this chain tight, i.e.

$$\bigcap_{p\ge 1} L^p = L^\infty \ ?$$

$\Longleftrightarrow$ is it true that a r.v. X whose all absolute moments are finite ($\mathbb{E}|X|^p < \infty$) must necessarily be bounded a.s.?

NO: $X \sim N(0,1)$ is a counterexample: $\mathbb{E}|X|^p = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |x|^p e^{-x^2/2}\,dx$ converges $\forall p \ge 1$

- How can we access the "gap" between L^p and L^∞?