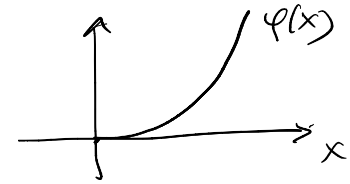
LECTURE 12

ORLICZ SPACES — more general than L^p.

Def. A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an Orlicz function if it is convex, increasing, and

$$\psi(x) \to \begin{cases} 0 & \text{as } x \to 0 \\ \infty & \text{as } x \to \infty \end{cases}$$



Ex. $\psi(x) = x^p$, $\psi(x) = e^x - 1$.

Better define unit ball first?

Def For a given Orlicz function ψ, the Orlicz norm of a r.v. X is defined as

$$\|X\|_\psi := \inf\{K > 0 : \mathbb{E}\,\psi(|X|/K) \leq 1\}$$

It is indeed a norm on the "Orlicz space"

$$L_\psi := \{\text{r.v.'s } X \text{ such that } \|X\|_\psi < \infty\}.$$

Ex: (a) $\psi(x) = x^p \Rightarrow L^p$

(b) $\psi(x) = e^x - 1 \Rightarrow L^p \supset \underbrace{L_\psi}_{\text{in the "gap"}} \supset L^\infty \quad \forall p \geq 1$

: $X \sim N(0,1)$ belongs to L_ψ.

SUBGAUSSIAN DISTRIBUTIONS.

- Recall Hoeffding's inequality (Lec. 6): if $X_i \sim \text{Sym Ber}$ iid, then

$$\mathbb{P}\left\{\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N} X_i\right| \geq t\right\} \leq 2\exp\left(-\frac{t^2}{2}\right) \qquad \forall t \geq 0.$$

↑gaussian tail

- Same holds for $X_i \sim N(0,1)$ and $X_i \sim \text{Unif}[-\frac{1}{2},\frac{1}{2}]$ (Ex)
- What is the biggest class of distributions of X_i that obey H.I.?
- For $N=1$, H.I becomes

$$\boxed{\mathbb{P}\{|X_i| \geq t\} \leq 2\exp(-t^2/2)} \quad (*)$$

$\Rightarrow$ X_i's must satisfy that.

- We call $\forall$ such X_i $\boxed{\text{subgaussian}}$ r.v's

& We will show that H.I. holds for such X_i. (PLAN)

$(*)$ = "X_i is as good as $N(0,1)$" in many ways.

- What do we know about $g \sim N(0,1)$?

① **Tails:** $\mathbb{P}\{|g| \geq t\} \leq 2\exp(-t^2/2) \quad \forall t \geq 0.$

② **Moments:**

$$\mathbb{E} g^p = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^p e^{-x^2/2} dx \overset{\substack{\text{if } p \text{ is even} \\ \downarrow}}{=} (p-1)!! \qquad \text{(by parts)}$$

$$= 1 \cdot 3 \cdot 5 \cdots (p-3)(p-1) \leq \underbrace{p \cdot p \cdot p \cdots p \cdot p}_{p/2} \leq p^{p/2}$$

(and if p odd, $\mathbb{E} g^p = 0$).

$$\Rightarrow \quad \|g\|_{L^p} = (\mathbb{E}|g|^p)^{1/p} \leq \sqrt{p} \qquad \forall p \geq 1.$$

(MUST grow to ∞)

③ **MGF:**
$$\mathbb{E} e^{\lambda g} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\lambda x} e^{x^2/2} dx = e^{\lambda^2/2} \qquad \forall \lambda \in \mathbb{R}$$

④ **MGF of g^2:**
$$\mathbb{E} e^{g^2/4} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{e^{x^2/4} e^{-x^2/2}}_{\substack{\| \\ e^{-x^2/4}}} dx \leq 2.$$

$\Rightarrow$ for Orlicz function

$\psi_2(x) := e^{x^2} - 1, \quad \mathbb{E}\psi_2(|g|/2) \leq 1 \quad \Rightarrow \quad \|g\|_{\psi_2} \leq 2.$

NEXT UP: $\forall$ distribution, ①–④ are equivalent, up to constants:

$\|$ subgauss. tails $\Leftrightarrow$ moments $O(\sqrt{p}) \Leftrightarrow$ MGF $e^{O(\lambda^2)} \Leftrightarrow \|X\|_{\psi_2} = O(1)$