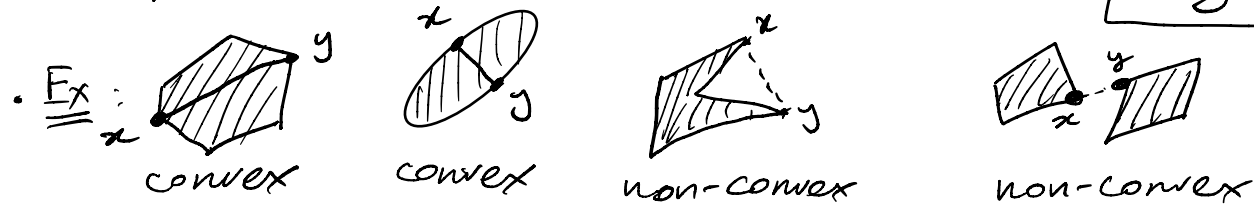


- One more example of how probability can help in high dim's: computational geometry

Def A set $T \subset \mathbb{R}^n$ is convex if $\forall x, y \in T, [x, y] \in T$

↑
segment 

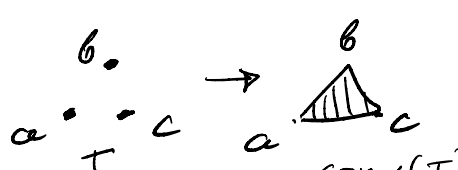


- How can one transform nonconvex set $\rightarrow$ convex?

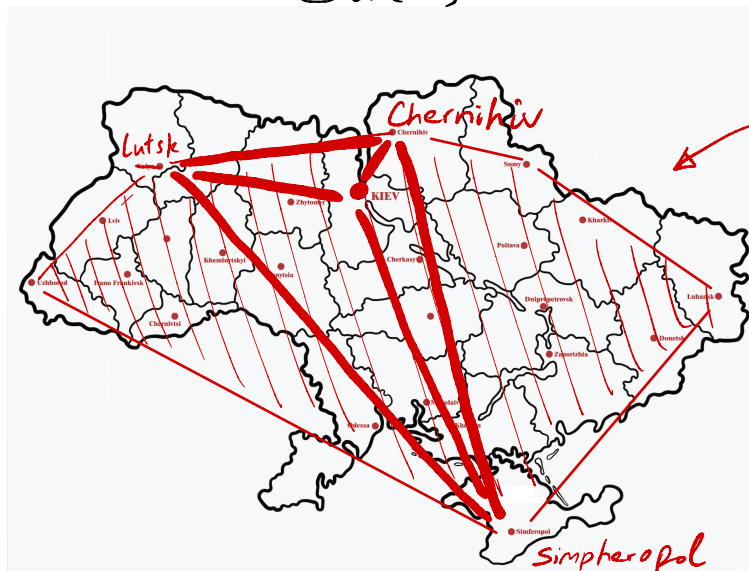
Def The convex hull of a set $T \subset \mathbb{R}^n$, denoted $\boxed{\text{conv}(T)}$, is the smallest convex set that contains T .

Ex: (a) 

(b) $T = \{a, b\}$ (two pts) $\Rightarrow \text{conv}(T) = [a, b]$ (segment)

(c) 

(d)



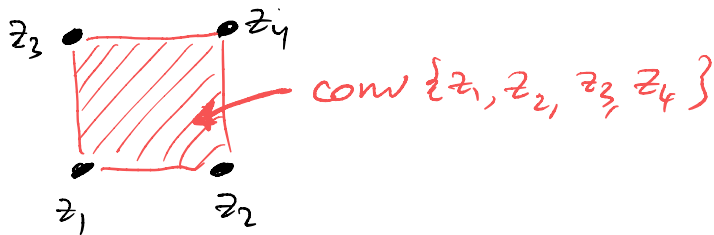
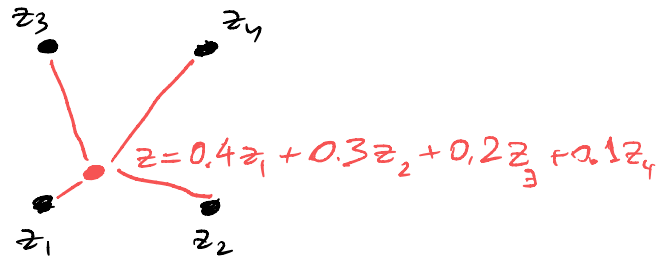
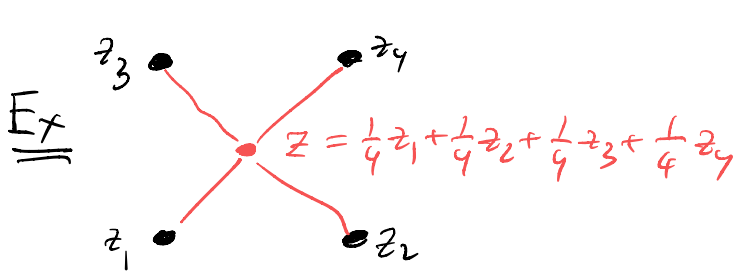
$T = \text{cities in Ukraine}$
 $\text{conv}(T)$

Fact $\forall z \in \text{conv}(T)$ can be expressed as

$$z = \sum_{i=1}^m \lambda_i z_i, \text{ where } \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1, z_i \in T \quad (*)$$

• PROOF: DIY (do it yourself).

• (*) is called a "convex combination". It is a linear combination, similar to a basis expansion of z , but non-unique (if $m > n$)



Q: Computing convex hull: how large is m ?

Caratheodory Thm $\forall$ point in $\text{conv}(T)$ can be expressed as a convex combination of $\leq n+1$ points from T .

Ex Kyiv $\in \text{conv}\{\text{Lvtsk, Chernihiv, Simpheropol}\}$

$n=2$.

Remark: $n+1$ is unimprovable (e.g. in the Kyiv example above)

But: if we allow to approximate x , then
a dramatic improvement!

Thm (Approx. Carathéodory) Let $T \subset \mathbb{R}^n$, $\text{diam}(T) \leq 1$.
Then $\forall x \in \text{conv}(T)$, $\forall k \in \mathbb{N} \quad \exists x_1, \dots, x_k \in T$:
$$\left\| x - \frac{1}{k} \sum_{i=1}^k x_i \right\|_2 \leq \frac{1}{\sqrt{2k}}$$

just scaling
Euclidean norm in $\mathbb{R}^n$

Remark: if we want the error $= \varepsilon (= \frac{1}{\sqrt{2k}})$ then $k = \frac{1}{2\varepsilon^2}$
Dimension-free! Does NOT depend on n ,
or geometry of T

PROOF by a probabilistic method: the "empirical method of Maurey"

WILL USE STANDARD FACTS OF PROBABILITY:

① Def of expectation of a discrete r.v. X :

if X takes values x_i with prob. p_i ,

$$\mathbb{E}X \stackrel{\text{def}}{=} \sum_i p_i x_i \quad \left(\begin{array}{l} \text{analogous to continuous case} \\ \text{where } \mathbb{E}X = \int_{-\infty}^{\infty} x p(x) dx \end{array} \right)$$

② $\text{Var}(X) = \mathbb{E}(X - \mathbb{E}X)^2 = \mathbb{E}(X^2) - (\mathbb{E}X)^2 \stackrel{(*)}{=} \frac{1}{2} \mathbb{E}(X - X')^2$

where X' is an independent copy of X

(i.e. X, X' are independent and have the same distribution)

Proof of (*):
$$\begin{aligned} \mathbb{E}(X - X')^2 &= \mathbb{E}[X^2 - 2XX' + (X')^2] \\ &= \mathbb{E}[X^2] - 2 \underbrace{\mathbb{E}(X)}_{\mathbb{E}X} \underbrace{\mathbb{E}(X')}_{\mathbb{E}X} + \underbrace{\mathbb{E}(X')^2}_{\mathbb{E}X^2} \quad \left(\begin{array}{l} \text{linearity \& indep.} \\ \text{same distr.} \end{array} \right) \\ &= 2 \left(\mathbb{E}(X^2) - (\mathbb{E}X)^2 \right) = 2 \text{Var}(X). \quad \text{QED} \end{aligned}$$

③ Extend (*) for random vectors X in $\mathbb{R}^n$:

$$\mathbb{E} \|X - \mathbb{E}X\|_2^2 = \frac{1}{2} \mathbb{E} \|X - X'\|_2^2 \quad \text{where } X' \text{ is an indep. copy of } X.$$

Proof of Approximate C.T.

- Fix $x \in \text{conv}(T)$, express it as a convex combination

$$x = \sum_{i=1}^m \lambda_i z_i, \quad \lambda_i \geq 0, \quad \sum_{i=1}^m \lambda_i = 1, \quad z_i \in T.$$

potentially large

- Interpret λ_i as probabilities, the sum as expectation

let r.v. vector Z take value z_i with prob. λ_i

$$\Rightarrow x = \mathbb{E} Z$$

- Consider indep. copies $z_1, z_2, \dots$ of Z .

$$\text{SLLN: } \frac{1}{k} \sum_{i=1}^k z_i \rightarrow \mathbb{E} Z = x \quad \text{a.s.}$$

- Error:

$$\mathbb{E} \left\| x - \frac{1}{k} \sum_{i=1}^k z_i \right\|_2^2 = \mathbb{E} \left\| \frac{1}{k} \sum_{i=1}^k (z_i - \underbrace{\mathbb{E} z_i}_{\mathbb{E} Z}) \right\|_2^2$$

$$= \frac{1}{k^2} \mathbb{E} \left\| \sum_{i=1}^k (z_i - \mathbb{E} z_i) \right\|_2^2$$

$$= \frac{1}{k^2} \sum_{i=1}^k \underbrace{\mathbb{E} \|z_i - \mathbb{E} z_i\|_2^2}_{\mathbb{E} \|Z - \mathbb{E} Z\|_2^2} \quad (\text{variance of sum: HW 1 Problem 5})$$

$$= \frac{1}{k} \mathbb{E} \|Z - \mathbb{E} Z\|_2^2 = \frac{1}{2k} \mathbb{E} \|Z - Z'\|_2^2 \quad (\text{fact ③ previous page})$$

$$\leq \frac{1}{2k} \quad \text{diam}(T) \leq 1 \text{ since } z, z' \in T$$

- $\Rightarrow \exists$ realization of the r.v.'s $z_1, \dots, z_k$ s.t.

$$\left\| x - \frac{1}{k} \sum_{i=1}^k z_i \right\|_2^2 \leq \frac{1}{2k} \quad \text{Since } z_i \in T, \quad \text{Q.E.D.}$$