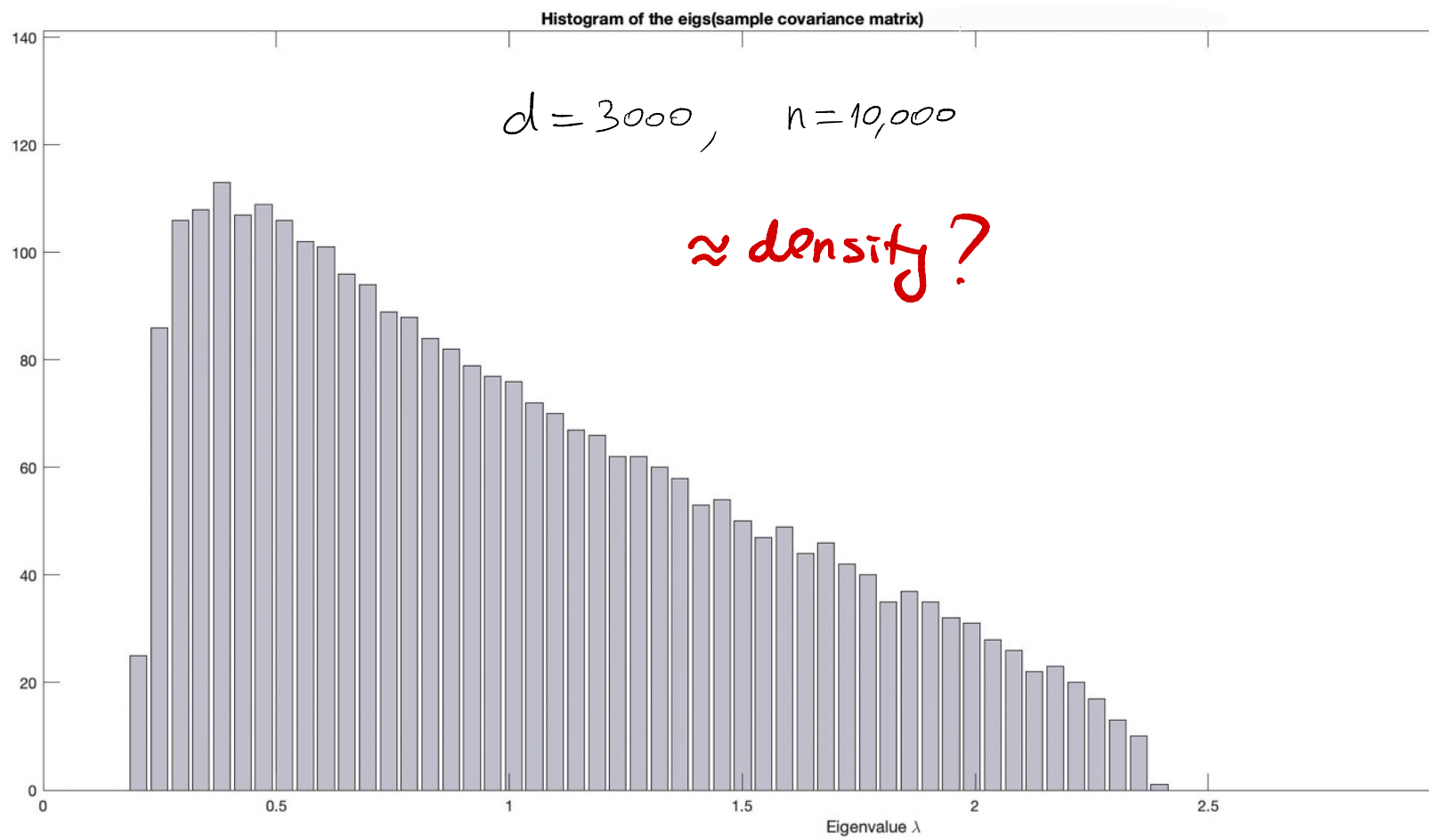
LECTURE 25

Last class An experiment: the spectrum of "white noise" data

$$X_1, \dots, X_n \sim N(0, I_d)$$

Sample cov. matrix: $\Sigma_n := \frac{1}{n} \sum_{k=1}^{n} X_k X_k^T$ symmetric $d \times d$ random matrix.

Eigenvalues:



$\Downarrow$ general

Q What do eigenvalues of a random matrix look like?

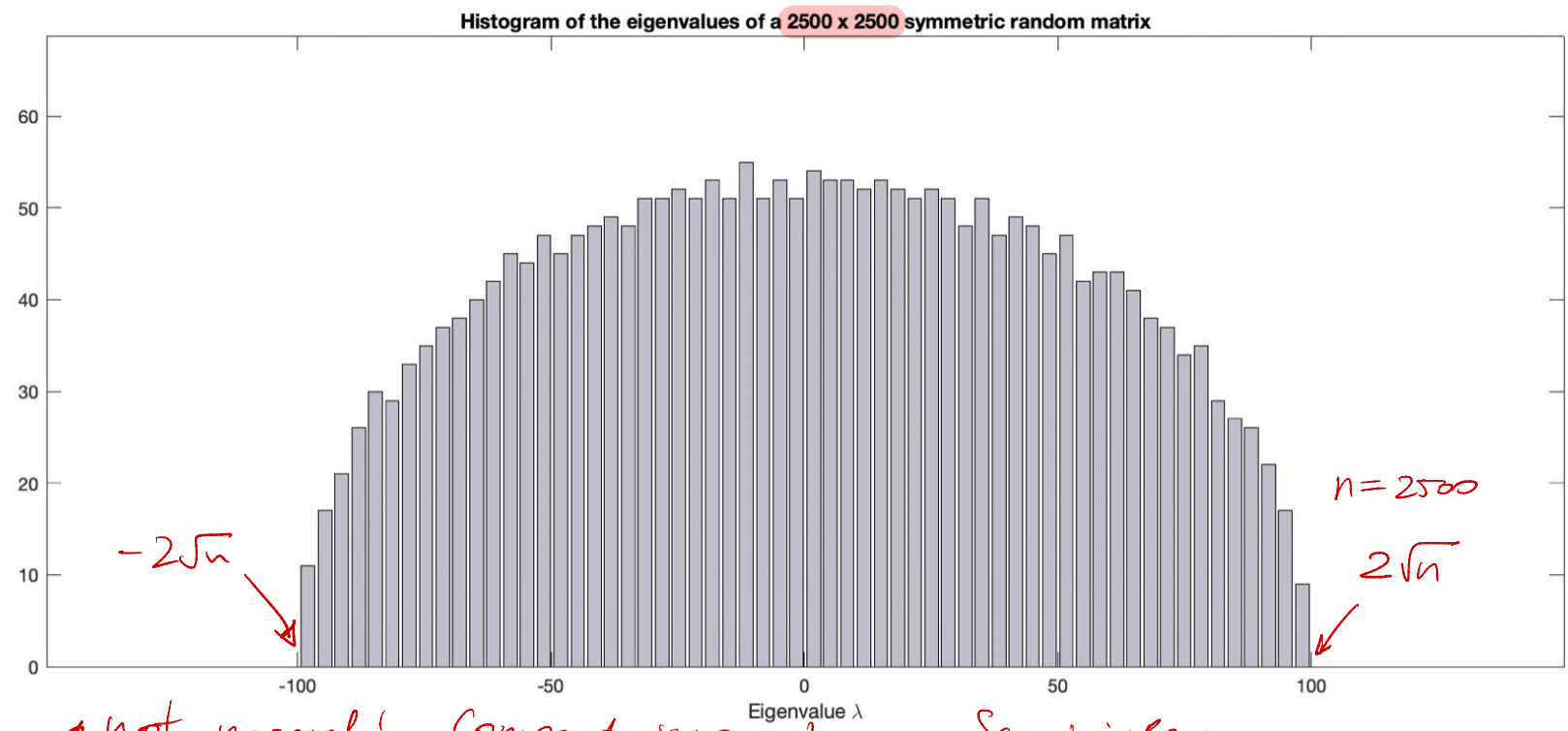
RANDOM MATRIX THEORY

Experiment Simpler model: $n \times n$ symmetric random matrix with independent entries, e.g. ± 1:

on and above diagonal

$$W = \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & - & - \\ - & + & - & + \end{bmatrix}$$

eigenvectors of W:



not normal! Compact support. Semicircle:

THM (Semicircle law, E. Wigner '1958) — informally:

Let W be an $n \times n$ symmetric random matrix whose entries on & above diagonal are iid. mean 0, var 1. Then, as $n \to \infty$, the histogram of the eigenvalues of $\frac{1}{\sqrt{n}} W$ converges to the "semicircle density"

$$p_{sc}(x) = \frac{1}{2\pi}\sqrt{4 - x^2} \quad \text{on } [-2, 2]$$

For the proof — BACKROUND IN LINEAR ALGEBRA :

① Matrix inversion :

(a) 2×2: $M = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ $\quad M^{-1} = \frac{1}{ad-b^2}\begin{bmatrix} d & -b \\ -b & a \end{bmatrix} \Rightarrow M^{-1}_{11} = \frac{d}{ad-b^2} = \frac{1}{a-b^2/d}$.

(b) $M = \left[\begin{array}{c|c} \overset{1}{a} & \overset{n-1}{b^T} \\ \hline b & D \end{array}\right] \begin{matrix} \}1 \\ \}n-1 \end{matrix}$ $\quad \boxed{M^{-1}_{11} = \frac{1}{a - b^T D^{-1} b}}$ (check! "Schur complement")

② Random quadratic form : if $g \sim N(0, I_n)$ then

$$\mathbb{E}\, g^T M g = \mathbb{E} \sum_{i,j=1}^{n} M_{ij} g_i g_j = \sum_{i,j=1}^{n} M_{ij} \mathbb{E} g_i g_j = \sum_{i=1}^{n} M_{ii} = \boxed{tr(M)}.$$

$\mathbb{E} g_i g_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$

③ $tr(M) = \sum_{i=1}^{n} M_{ii} = \sum_{k=1}^{n} \lambda_k(M)$

④ Cyclic property: $tr(AB) = tr(BA)$

⑤ $\lambda_i(M^{-1}) = \frac{1}{\lambda_i(M)}$; $\lambda_i(M - zI) = \lambda_i(M) - z$.

TOWARD THE PROOF OF SEMICIRCLE LAW:

- Will observe eigenvalues λ_i of a matrix A indirectly, with a function ("observable").

- Ex (a) $f(z) = \det(A - zI)$ characteristic polynomial
  $(\lambda_i) =$ roots of $f(z)$. More convenient is

  (b) $f(z) = \operatorname{tr} \underbrace{(A - zI)^{-1}}_{\text{resolvent}} \overset{p.3}{=} \sum \operatorname{eigs}(A - zI)^{-1} \overset{p.3}{=} \sum_{i=1}^{n} \frac{1}{\lambda_i - z}$

  $(\lambda_i) =$ poles of $f(z)$

- Normalize: $\frac{1}{n} \operatorname{tr}(A - zI)^{-1} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\lambda_i - z}$ $\xrightarrow{n \to \infty} \mathbb{E}$ $\Rightarrow$ probabilistic view:

Def The Stieltjes transform of the distribution of a r.v. X is

$$\boxed{s(z) := \mathbb{E}\left[\frac{1}{X - z}\right], \quad z \in \mathbb{C}}$$

Ex (a) $X \sim \operatorname{Unif}\{\lambda_1, \dots, \lambda_n\} \Rightarrow s(z) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\lambda_i - z}$.

(b) X continuous, pdf $\rho \Rightarrow s(z) = \int_{-\infty}^{\infty} \frac{\rho(x)}{x - z} \, dx$ (~ Residue)

(c) $\rho = \rho_{sc}$ (semicircle, p. 2) $\Rightarrow s(z) = \frac{1}{2\pi} \int_{-2}^{2} \frac{\sqrt{4 - x^2}}{x - z} \, dx = \boxed{\frac{-z + \sqrt{z^2 - 4}}{2}}$ (DIY)

FACT Stieltjes transform determines the distribution of X uniquely.

In particular, if $X \sim \operatorname{Unif}\{x_1, \dots, x_n\}$, $(x_i) =$ poles of $s(z)$

If X has pdf $\rho \Rightarrow \rho(x) = \lim_{\varepsilon \to 0^+} \frac{s(x - i\varepsilon) - s(x + i\varepsilon)}{2i\pi}$ (DIY)

"Stieltjes inversion"

PROOF OF SEMICIRCLE LAW (informal)

according to T. Tao, V. Vu, [Random matrices: universality of local eigenvalue statistics '2009 p.50]

- Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $\frac{1}{\sqrt{n}} W$.

Stieltjes transform of $\mathrm{Unif}\{\lambda_1, \dots, \lambda_n\}$ is

$$s_n(z) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\lambda_i - z} = \frac{1}{n} \sum_{i=1}^{n} \mathrm{eig}_i\left(\frac{1}{\sqrt{n}} W - z\right)^{-1} = \frac{1}{n} \mathrm{tr}\Big(\underbrace{\frac{1}{\sqrt{n}} W - I}_{\overset{\shortparallel}{M}}\Big)^{-1}$$

$$= \frac{1}{n} \sum_{k=1}^{n} M^{-1}_{kk} \qquad (*)$$

?

- $$W = \left[\begin{array}{c|c} W_{11} & g^T \\ \hline g & \bar{W} \end{array}\right] \Rightarrow M = \frac{1}{\sqrt{n}} W - zI = \begin{array}{c} \begin{array}{cc} 1 & n-1 \end{array} \\ \left[\begin{array}{c|c} \frac{W_{11}}{\sqrt{n}} - z & \frac{1}{\sqrt{n}} g^T \\ \hline \frac{1}{\sqrt{n}} g & \bar{M} \end{array}\right] \end{array} \begin{array}{l} 1 \\ n-1 \end{array}$$

Matrix inversion (p. 3) $\Rightarrow$

$$M^{-1}_{11} = \frac{1}{\underbrace{\frac{W_{11}}{\sqrt{n}}}_{\downarrow \text{ as } n \to \infty \; 0} - z - \underbrace{\frac{1}{n} g^T \bar{M}^{-1} g}_{\text{concentration}}}$$

g is independent of $\bar{M}$!

$$\mathbb{E} \frac{1}{n} g^T \bar{M}^{-1} g = \frac{1}{n} \mathrm{tr}(\bar{M}^{-1}) \quad \text{(random quadratic form p. 3)}$$

$$\approx \frac{1}{n} \mathrm{tr}(M^{-1}) \quad \text{(change from } n-1 \to n \text{ is small)}$$

$$= s_n(z).$$

$$\Rightarrow M^{-1}_{11} \approx \frac{1}{-z - s_n(z)}. \qquad \text{Similarly for } M^{-1}_{kk}.$$

- $$\overset{(*\,p.1)}{\Longrightarrow} s_n(z) \approx \frac{1}{-z - s_n(z)}. \quad \overset{\text{solve}}{\Longrightarrow} s(z) \approx \frac{-z + \sqrt{z^2 - 4}}{2}.$$

$\Rightarrow$ Same as Stieltjes transform of p_{sc} ((c) p. 3) ☺

Uniqueness $\Rightarrow$ QED