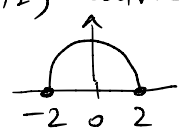
LECTURE 27

THE FUNDAMENTAL LAWS OF RANDOM MATRIX THEORY

Last 2 classes:

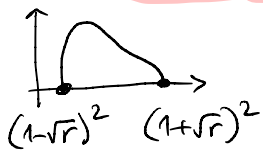
① **Semicircle law** [Wigner 1959]

W: $n \times n$ symmetric, iid $N(0,1)$ entries ("Wigner")

bulk of eigs$\left(\frac{1}{\sqrt{n}} W\right) \longrightarrow$ 

② **Marchenko–Pastur law** [1967]

$\Sigma_n = \frac{1}{n} \sum_{i=1}^{n} X_i X_i^T$, $X_i \sim N(0, I_d)$ ("Wishart")

bulk of eigs$(\Sigma_n) \longrightarrow$ 

where $r = d/n$

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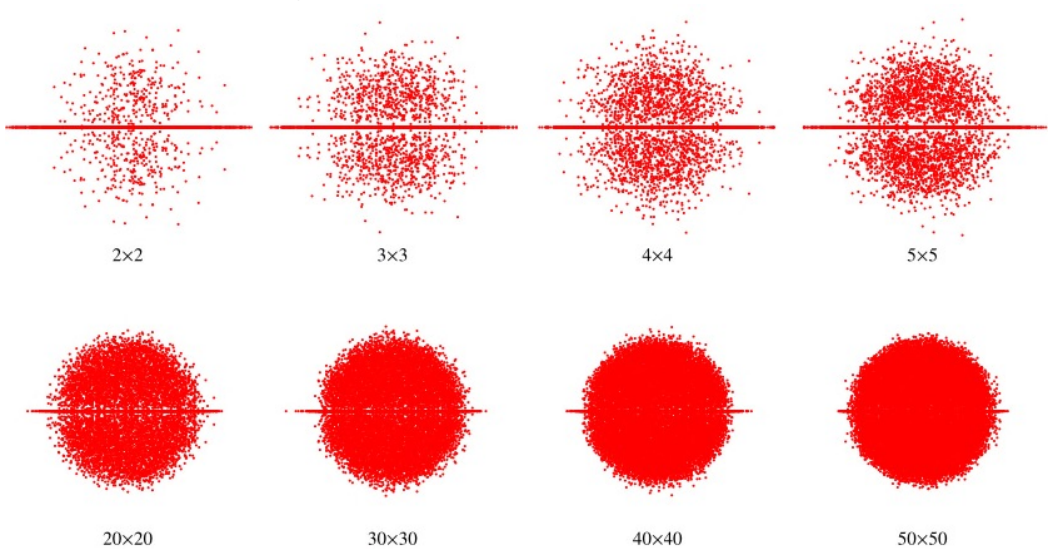
③ **Circular law** [Ginibre 1965]

$M = n \times n$ iid $N(0,1)$ entries (No symmetry) ("Ginibre")

bulk of eigs$\left(\frac{1}{\sqrt{n}} M\right) \to$ Unif(unit disc)



Rigorously: $\forall$ $B \subset \mathbb{C}$ Borel:

$$\lim_{n\to\infty} \frac{\#\{i : \lambda_i(M/\sqrt{n}) \in B\}}{n} = \frac{\text{Area}(B)}{\pi}.$$

- These "bulk" results do NOT say anything about:

EXTREME EIGENVECTORS.

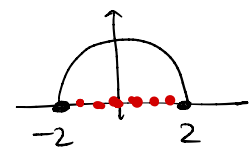
④ Bai-Yin law [1988] for Wigner matrix W:

i.e. with probability 1

$$\frac{\lambda_1(W)}{\sqrt{n}} \to 2 \quad \text{almost surely}$$

$\Rightarrow$ by symmetry,

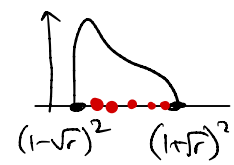
$$\frac{\lambda_n(W)}{\sqrt{n}} \to -2 \quad \text{almost surely}$$



⑤ Bai-Yin-Krishnaiah [1988] for Wishart (sample cov.) Σ_n:

$$\lambda_1(\Sigma_n) \to (1+\sqrt{r})^2 \quad \text{almost surely}$$

$$\lambda_d(\Sigma_n) \to (1-\sqrt{r})^2 \quad \text{almost surely}$$



- Fluctuations of λ_1, λ_d? Distribution?

⑥ Tracy-Widom law [1994]

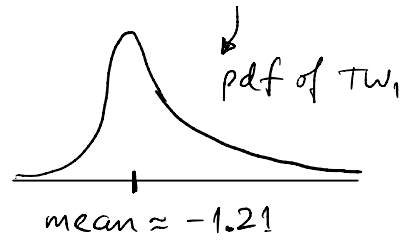
$\ll 1$?! Rigid

(a) for Wigner matrices: $\lambda_1(W) \approx 2\sqrt{n} \pm O(n^{-1/6})$.

More precisely:

$$(\lambda_1(W) - 2\sqrt{n})\, n^{1/6} \to TW_1 \quad \text{(Tracy-Widom distribution)}$$

By symmetry, same holds for $-\lambda_n(W)$



(b) for Wishart matrices [Johnstone 2001, Feldheim-Sodin 2009]:

$$\lambda_1(\Sigma_n) \approx (1+\sqrt{r})^2 \pm O(d^{-1/6}), \quad \lambda_d(\Sigma_n) \approx (1-\sqrt{r})^2 \pm O(d^{-1/6})$$

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More precisely:

$$\frac{\lambda_1(\Sigma_n) - (1+\sqrt{r})^2}{(1+\sqrt{r})^{2/3}\, d^{1/6}} \to TW_1, \quad \frac{\lambda_d(\Sigma_n) - (1-\sqrt{r})^2}{(1-\sqrt{r})^{2/3}\, d^{1/6}} \to TW_1,$$

- All of this was for "pure noise" models.

What about structured data?

⑦ Spike models [Baik–Ben Arous–Péché 2005]

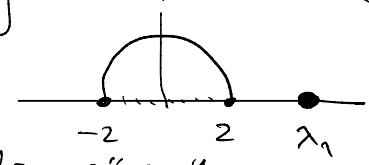
see [Bandeira–Singer–Strohmer]

(a) Wigner W

$$M := \underbrace{\tfrac{1}{\sqrt{n}} W}_{\text{"noise"}} + \underbrace{\beta u u^T}_{\text{"signal"}} \quad \text{where } u \in \mathbb{R}^d \text{ is unit} \leftarrow \text{fixed}$$

$\beta > 0$ is signal-to-noise ratio (SNR)

Then, as $n \to \infty$, almost surely:

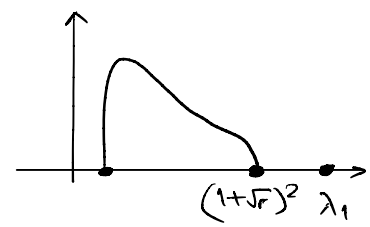


- $\lambda_1(M) \longrightarrow \begin{cases} 2 & \text{if } \beta \le 1 \\ \beta + \frac{1}{\beta} & \text{if } \beta \ge 1 \end{cases}$ signal "drowns" in the noise / signal is detectable

Fluctuations of $\lambda_1(M) \overset{\text{dist}}{=\!=} \text{Tracy–Widom}$.

- $\langle \underset{\text{eigenvector}}{v_1(M)}, u \rangle^2 \to \begin{cases} 0 & \text{if } \beta \le 1 \\ 1 - \frac{1}{\beta^2} & \text{if } \beta \ge 1 \end{cases}$ no correlation /

(b) Wishart: assume $\Sigma = \mathbb{E} X_i X_i^T = I + \beta u u^T$.



$$\lambda_1(M) \xrightarrow{\text{a.s.}} \begin{cases} (1+\sqrt{r})^2 & \text{if } \beta \le \sqrt{r} \\ (\beta+1)(1+\frac{r}{\beta}) & \text{if } \beta \ge \sqrt{r} \end{cases}$$

signal "drowns" in the noise / signal is detectable

$$\langle v_1(\Sigma_n), u \rangle^2 \xrightarrow{\text{a.s.}} \begin{cases} 0 & \text{if } \beta \le \sqrt{r} \\ \frac{1 - r/\beta^2}{1 + r/\beta^2} & \text{if } \beta \ge \sqrt{r} \end{cases}$$

no correlation /

$\Rightarrow$ PCA works iff $\beta \ge \sqrt{r} = \sqrt{\frac{d}{n}} \iff \boxed{n \ge d/\beta^2}$

8 Joint eigenvalue distribution

For Wigner $W = \frac{A+A^T}{2}$ where $A: d\times d$ with iid $N(0,1)$ entries

($\Rightarrow W$ has $N(0,1/2)$ off-diagonal entries and $N(0,1)$ on the diagonal)

joint density of $\lambda_1(W) \geq \lambda_2(W) \geq \dots \geq \lambda_n(W)$:

$$p(x_1,\dots,x_n) \propto \underbrace{\exp\left(-\frac{1}{2}\sum_{i=1}^n x_i^2\right)}_{\text{as for independent } N(0,1) \text{ r.v's}} \underbrace{\prod_{i<j} |x_i - x_j|}_{\text{Vandermonde determinant: REPULSION}}$$

9 Universality

All the results above (except 8) hold not only for normal but for more general distributions (mean 0, var 1) — NO CHANGE

10 Connections

(a) Physics: "Wigner's surmise":

distr. of energy levels of heavy nuclei

$\approx$ distr. of eigenvalues of Wigner random matrix.

Repulsion!

(b) Number theory: "Montgomery's Pair Correlation Conjecture" (MPCC)

for Riemann zeta function

$$\zeta(s) := \sum_{n\geq 1}^{\infty} n^{-s} \overset{\text{Euler}}{=\!=} \prod_{p:\text{prime}} \frac{1}{1-p^{-s}}$$

(defined on $Re(s)>1$, analytically continued to $\mathbb{C}$)

Riemann hypothesis: zeros of $\zeta(s) \in -2\mathbb{Z} \cup \{s : Re(s) = 1/2\}$

MPCC: distribution of spacings between the zeros

$\approx$ distr. of spacings between eigs (Wigner random matrix).

$\Rightarrow$ Repulsion