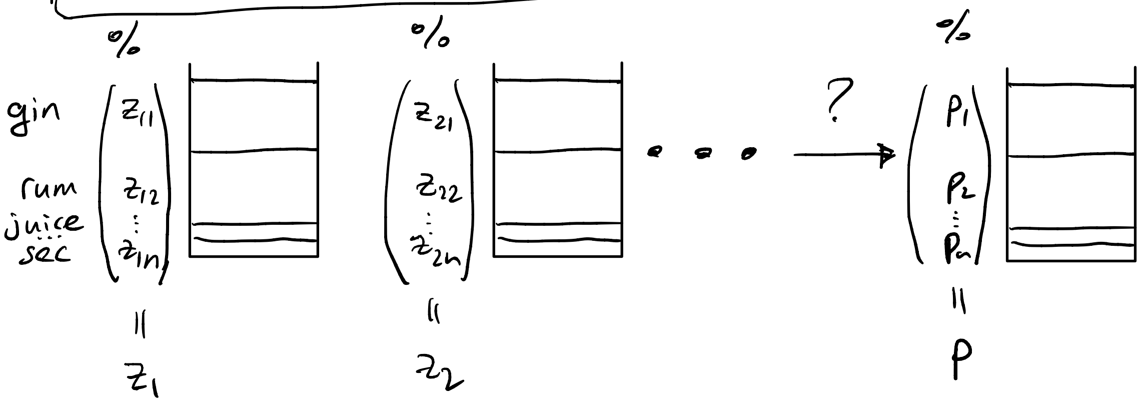
LECTURE 3

Application of Approx. Caratheodory Thm:

Cocktail Problem You are given N glasses with different cocktails, each made by mixing n ingredients in different proportions. Make a glass of cocktail with given proportions $p_1, \dots, p_n$.



Equivalently, we need to find λ_1 (portion of glass 1), λ_2 (port. of glass 2)

such that: $$p = \sum_{i=1}^{N} \lambda_i z_i, \quad \lambda_i \geq 0, \quad \sum_{i=1}^{N} \lambda_i = 1$$

proportions need to be nonnegative — and sum to 100%.

- i.e. need to express z as a convex combination of vectors $z_1, \dots, z_N$
- "Convex program" finds a solution in polynomial time.
- Approx. Caratheodory Thm transforms it into an approximate solution with few glasses, mixed in equal proportions; a fast randomized alg.

Relevance of cocktail problem

(a) (Portfolio Building)

ingredients = stocks
glasses of cocktails = mutual funds
empty glass = portfolio

Problem: create a new mutual fund with a given combination of stocks by combining the mutual funds that are available on the market.

Solution: as above – a fast randomized algorithm builds a portfolio with few mutual funds.

(b) (Factor analysis)

$z_1, \dots, z_N$ = a dictionary of factors that need to explain z = behavior (consumer, animal, etc.)

Sol: behavior is explained by few factors.

$$z = \sum \lambda_i z_i \Rightarrow \text{factor 1 explains } \lambda_i \% \text{ of behavior, } \dots$$

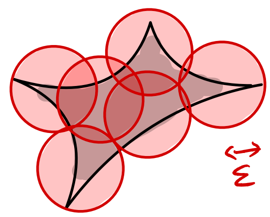
$\Rightarrow$ A PARSIMONIOUS MODEL.

One more application of A.C.T:

Covering Numbers.

Def The covering number of a set $T \subset \mathbb{R}^n$ at scale $\varepsilon > 0$ is the smallest number of Euclidean balls of radius ε needed to cover T. Denoted $N(T, \varepsilon)$.

Ex

T 

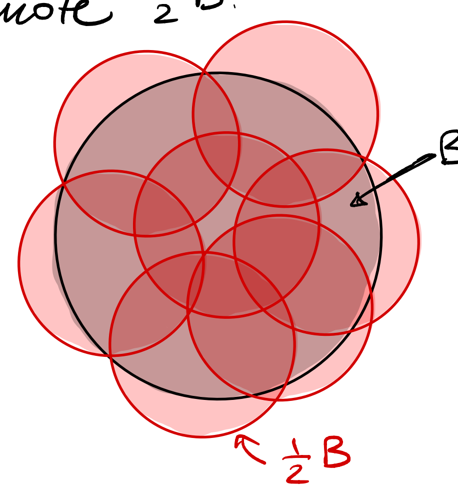
$N(T,\varepsilon) \le 6$

- Covering numbers is a measure of complexity of T (like area, volume ...)
- Suffer from the curse of high dimensionality:

Prop For B = unit Euclidean ball, we have
$$N(B, \tfrac{1}{2}) \ge 2^n$$

Exponential!

Proof Assume B can be covered by N copies of a ball twice smaller, which we denote $\frac{1}{2}B$.
Comparing the volumes gives



$$\mathrm{Vol}(B) \le N \cdot \underbrace{\mathrm{Vol}(\tfrac{1}{2}B)}_{= \left(\frac{1}{2}\right)^n \mathrm{Vol}(B)}.$$

$\Rightarrow N \ge 2^n$. QED.

- Generally, covering #s are exponential in the dimension, But not always!

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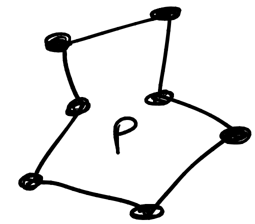
THM Let P be a polytope in $\mathbb{R}^n$ with m vertices, $\operatorname{diam}(P) \le 1$.
Then $N(P, \varepsilon) \le m^{\frac{1}{2\varepsilon^2}}$.

Dimension-free.
Polynomial in N



Proof

- $P \subset \operatorname{conv}(T)$ where $T = \{\text{vertices of } P\}$
- Approx. Caratheodory Thm states:
  $\forall x \in P \subset \operatorname{conv}(T)$ is within distance $\frac{1}{\sqrt{2k}}$ from some point in the set
  $$\mathcal{N} := \left\{ \frac{1}{k} \sum_{i=1}^{k} z_i \; : \; z_i \in T \right\}$$

$\Rightarrow$ $\forall x \in P$ is covered by a ball of radius $\frac{1}{\sqrt{2k}}$ and center $\in \mathcal{N}$.

$\Rightarrow$ $N\left(P, \frac{1}{\sqrt{2k}}\right) \le |\mathcal{N}| \le m^k$

\# ways to choose k elements z_i from the set T of m elements, with repetition

- Choose k: $\frac{1}{\sqrt{2k}} = \varepsilon$ $\left(k = \frac{1}{2\varepsilon^2} \right)$ $\Rightarrow$ QED.

Small covering # $\Rightarrow$ small volume