**THEM (Volume of polytopes) [Carl-Pajor '88]**

Let $B = $ unit Euclidean ball, $P \subset B$ polytope with $m$ vertices. Then

$$\frac{\text{Vol}(P)}{\text{Vol}(B)} \leq \left(4 \sqrt[n]{\frac{\log m}{n}}\right)^n.$$ 

Exponentially small unless $m > \exp(cn)$ vertices.

**Proof.** By def. of covering numbers, $P$ can be covered by $N(P, \varepsilon)$ copies of a ball of radius $\varepsilon$, denoted $\varepsilon B$.

Comparing the volumes yields

$$\text{Vol}(P) \leq N(P, \varepsilon) \cdot \text{Vol}(\varepsilon B).$$

- By Thm. 4 (lec 3), $N(P, \varepsilon) \leq M \frac{\text{diam}(P)}{2\varepsilon^2} \leq m^{\frac{\text{diam}(P)}{\varepsilon^2}}$.

- By volume scaling, $\text{Vol}(\varepsilon B) = \varepsilon^n \cdot \text{Vol}(B)$.

Substitute $\Rightarrow$

$$\frac{\text{Vol}(P)}{\text{Vol}(B)} \leq M \varepsilon^n.$$

- Minimize RHS in $\varepsilon \Rightarrow$ QED. (DIY: take logs & differentiate)
Remarks

1. [Carl-Pajor] proved a slightly sharper bound \( (C \sqrt{\frac{\log(m/n)}{n}}) \).

2. It is optimal, attained by a random polytope
   \[ P = \text{conv} \{ x_1, \ldots, x_m \} \]
   where \( x_i = \text{indep. random pts on the unit sphere} \).
   [Dafnis-Giannopoulos-Tsolomitis '2003, 2009]

3. Hyperbolic intuition
   for \( \delta = \sqrt{\frac{\log n}{n}} \), Thu says: tiny ball!

   \[
   \text{Vol}(P) \leq \delta^n \, \text{Vol}(B) = \text{Vol}(\delta B).
   \]
   Thus the picture "actually" looks like this, even though it violates convexity.

4. "Hyperbolic intuition"

4. "Blind spider" experiment:
   - A random walk will stay in the "core" \( SB \),
     will not find the "tentacles".
   - The spider can't tell the polytope from the ball \( SB \).
   - Bad for algorithms (MCMC)


**Concentration Inequalities.**

- *X \approx \mathcal{E}X* with high probability.
  - closer to 1 than you think.

- Ex: normal distribution. \( X \sim N(\mu, \sigma^2) \) satisfies
  - \(|X - \mu| \leq 3\sigma\) with prob. 0.9973
    (see 68-95-99.7 rule)

A general tail bound:

\[
\Pr(g \geq t) \leq \frac{1}{t \sqrt{2\pi}} e^{-\frac{t^2}{2}} \quad \forall t \geq 0
\]

Proof

\[
\Pr(X \in A) = \int_A \rho(x) \, dx \quad \text{(density of } X)\]

\[
\Pr(g \geq t) = \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx.
\]

Not computable analytically.

To estimate, change variables \( x = t + y \Rightarrow \frac{x^2}{2} = \frac{t^2}{2} + ty + \frac{y^2}{2} \)

\[
\Pr(g \geq t) = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2} - ty} e^{-\frac{y^2}{2}} \, dy
\]

\[
\leq \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \int_0^\infty e^{-ty} \, dy.
\]

\[\leq \frac{1}{t\sqrt{2\pi}} e^{-\frac{t^2}{2}} \quad \text{``1/t (DIY)'', QED.}\]

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- By symmetry, 
\[ P\{ |g| \geq t \} = 2 \cdot P\{ g \geq t \} \leq \frac{1}{t} \sqrt{\frac{2}{\pi}} e^{-\frac{t^2}{2}} \] (*)&

- More generally, if \( X \sim N(\mu, \sigma^2) \) \( \Rightarrow X = \mu + \sigma g \)

\[ \Rightarrow P\{ |X-\mu| \geq t\sigma \} = P\{ |g| \geq t \} \leq e^{-\frac{t^2}{2}} \quad \forall t \geq 1. \]

*Ex: \( t=3 : \) \[ P\{ |X-\mu| \leq 3\sigma \} \geq 1 - \frac{1}{2} \sqrt{\frac{2}{\pi}} e^{-3\frac{3}{2}} \geq 0.997 \]

Remark: a simpler form of (*) often suffices.

Since \( \sqrt{\frac{2}{\pi}} \leq 1 \), we have:

\[
P\{ |g| \geq t \} \leq e^{-\frac{t^2}{2}} \quad \forall t \geq 1
\]

"Gaussian tail bound"